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LCR-structures and LCR-algebras

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Thesis submitted for the degree of "Doctor Philosophiæ"

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A Marilena.
Nel giorno della sua partenza.
Buon viaggio.

Il presente lavoro costituisce la tesi presentata da Daniele Gouthier, sotto la direzione del Prof. Giuseppe Tomassini, al fine di ottenere il diploma di *"Doctor Philosophiæ"* presso la Scuola Internazionale Superiore di Studi Avanzati, Classe di Matematica, Settore di Analisi Funzionale ed Applicazioni.

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Contents

Preface.

Let $\mathfrak{g}_0$ be a real Lie-algebra. A complex structure on $\mathfrak{g}_0$ is an endomorphism $J \in GL(\mathfrak{g}_0)$ such that $J^2 = -id$ and $[JX, JY] = [X, Y] + J[X, JY] + J[JX, Y]$, for all $X, Y \in \mathfrak{g}_0$, [JA]. If $\mathfrak{g}$ denotes the complexification of $\mathfrak{g}_0$, $\mathfrak{g} \doteq \mathfrak{g}_0 \otimes_{\mathbf{R}} \mathbf{C}$, then $\mathfrak{q} \doteq \{X - iJX : X \in \mathfrak{g}_0\}$ is a complex subalgebra and there is the vector space decomposition $\mathfrak{g} = \mathfrak{q} \oplus \overline{\mathfrak{q}}$. Conversely, any such splitting $\mathfrak{q} \oplus \overline{\mathfrak{q}}$ defines a complex structure on $\mathfrak{g}_0$ setting $JX = -Y$, if $X + iY \in \mathfrak{q}$.

A complex structure on $\mathfrak{g}_0$ induces a complex structure on G_0, the Lie-group associated to $\mathfrak{g}_0$, for which left translations are holomorphic.

The study of complex structure on even dimensional real Lie-algebras goes back to Morimoto, who showed that every reductive real Lie-algebra has infinitely many complex structures, [MO]. In [SN], D.Snow gave a complete classification of those complex structures on a reductive Lie-algebra, which are "regular" (see Introduction to Chapter 2).

A natural generalization of these complex structures is the notion of *CR-structure* which has been introduced in [GT] (see also [AHR]). A CR-structure on a real Lie-algebra $\mathfrak{g}_0$ is the datum of a pair $(\mathfrak{p}, J)$, where $\mathbf{p}$ is a real subspace of $\mathbf{g}_0$ and $J \in GL(\mathbf{p})$ satisfies

1. $J^2 = -id$;
2. $[JX, JY] = [X, Y] + J[X, JY] + J[JX, Y], \forall X, Y \in \mathbf{p}$;
3. $[JX, JY] - [X, Y] \in \mathbf{p}, \forall X, Y \in \mathbf{p}$.

Even in the present case, the complex subspace $\mathbf{q} = \{X - iJX : X \in \mathbf{p}\}$ is a subalgebra of $\mathbf{g}$ such that $\mathbf{q} \cap \overline{\mathbf{q}} = \{0\}$, in such a way that $\mathbf{g} = \mathbf{q} \oplus \overline{\mathbf{q}} \oplus V$, where V is a linear space spanned by real vectors. Both the notations, $(\mathbf{p}, J)$ and $\mathbf{q}$, are employed to indicate a CR-structure.

Consider now a real Lie-group G_0, whose Lie-algebra $Lie(G_0)$ is $\mathbf{g}_0$, endowed with a CR-structure. Then, the group G_0 inherits a structure of CR-manifold for which the left translations are CR-maps, [BOG], [WE], [AHR]. Moreover, if the CR-structure is such that $\mathbf{p}$ is a real subalgebra (and consequentely $\mathbf{q} \oplus \overline{\mathbf{q}}$ is a complex subalgebra of $\mathbf{g}$), the Lie-group G_0 is a Levi-flat manifold: i.e. foliated by complex submanifolds ([BOG]). In such a situation the CR-structure $(\mathbf{p}, J)$ is said to be *Levi-flat*. An interesting class of such CR-structures is given by the ones whose leaf through the unit of G_0 is a subgroup. A direct consequence of this fact is that both right and left translations are CR-maps. In particular, $\mathbf{p}$ is a real ideal of $\mathbf{g}_0$, ad_X is a CR-map, for every $X \in \mathbf{g}_0$, and the corresponding complex subalgebra $\mathbf{q}$ is an ideal. These CR-structures are said to be *CR-structure of Lie*. They are shortly called *LCR-structures*.

Via the knowledge of the LCR-structures is possible to study the Levi-flat ones. Indeed, consider the bilinear skewsymmetric form $\Gamma : \mathbf{p} \times \mathbf{p} \to \mathbf{p} : (X, Y) \mapsto [X, Y] - [JX, JY]$. The pair $(\mathbf{p}, \Gamma)$ is a Lie-algebra and the map J is invariant under Γ_X. Thus, any CR-structure $(\mathfrak{p}, J)$ on $\mathfrak{g}_0$ is a biinvariant structure on $(\mathfrak{p}, \Gamma)$, (see Chapter 2).

The content of this thesis is a general treatment of LCR-structures $(\mathfrak{p}, J)$ on a real Lie-algebra $\mathfrak{g}_0$. For our study, we adopt two points of view. According to the first one, the central role is taken by the pair $(\mathfrak{p}, J)$. We investigate the structure of the ideal $\mathfrak{p}$ and all the possible J's on it. Some limitations are found (semisimple compact Lie-algebras do not admit any LCR-structure) and a constructive method is developed (the LCR-structures of a solvable Lie-algebra are given on the even-dimensional ideals by the "multiplication by i"). The main result is a structure theorem for $(\mathfrak{p}, J)$, (Theorem 2.4.3):

let $\mathfrak{g}_0 = \mathfrak{r} \oplus_{ad} \mathfrak{s}$ *be a real Lie-algebra. Suppose* $(\mathfrak{p}, J)$ *is a LCR-structure on* $\mathfrak{g}_0$*; then* $(\mathfrak{p}_\mathfrak{r}, J_\mathfrak{r})$ *and* $(\mathfrak{p}_\mathfrak{s}, J_\mathfrak{s})$ *are LCR-structures on* $\mathfrak{r}$ *and* $\mathfrak{s}$*, respectively; and* $(\mathfrak{p}, J)$ *is their semidirect sum by the adjoint derivation. Vice versa, if one considers two LCR-structures* $(\mathfrak{p}_\mathfrak{r}, A)$ *and* $(\mathfrak{p}_\mathfrak{s}, D)$ *which verify*

1) $[\mathfrak{p}_\mathfrak{s}, \mathfrak{r}] \subset \mathfrak{p}_\mathfrak{r}$

2) $[\mathfrak{p}_\mathfrak{r}, \mathfrak{s}] \subset \mathfrak{p}_\mathfrak{r}$

3) $A[X, V] = [X, AV]$

4) $A[U, Y] = [U, DY]$

their semidirect sum by ad is a LCR-structure on $\mathfrak{g}_0$. ■

For the second approach we study the "CR-properties" of $\mathfrak{g}_0$ depending on a fixed LCR-structure $(\mathfrak{p}, J)$. As in the classical case, we introduce the fundamental notions of CR-nilpotence, CR-solvability, CR-semisemplicity. The characterization of these properties for a LCR-algebra are expressed, in terms of $\mathbf{g} = \mathbf{g}_0 \otimes_{\mathbf{R}} \mathbf{C}$ by the following table

nilpotent : $\mathcal{C}^k\mathbf{g} = 0$	CR-nilpotent : $\mathbf{q} \cap \mathcal{C}^k\mathbf{g} = 0$
solvable : $\mathcal{D}^k\mathbf{g} = 0$	CR-solvable : $\mathbf{q} \cap \mathcal{D}^k\mathbf{g} = 0$
semisimple : $B \neq 0$	CR-semisimple : $B_{\mathbf{q}} \neq 0$

(here, as usual $\mathcal{C}^k$ denotes the k^{th}-central element, $\mathcal{D}^k$ the k^{th}-derived and B the Killing form).

Furthermore, for a LCR-algebra a Levi-Mal'cev CR-decomposition is proved (Theorem 3.8.6): $\mathbf{g}$ is the semidirect sum by ad of a CR-solvable LCR-ideal and of a CR-semisimple sub-LCR-algebra.

As it is well known, reductive Lie-algebras have a central position in the theory of complex and CR-structures, [MO], [SN], [GT]. Indeed, Morimoto showed that they are always endowed with a complex structure, whenever they are even-dimensional and Snow classified their "regular" complex structures. In Snow's paper the regularity is given demanding the invariance of $\mathbf{q}$ under $ad_{\mathbf{h}}$, where $\mathbf{h}$ is a suitable Cartan subalgebra. In that situation, if Δ is the corresponding root set, then the complex structure $\mathbf{q}$ is given by

$$\mathbf{q} = \mathbf{q} \cap \mathbf{h} \oplus \oplus_{\alpha \in \Pi} \mathbf{g}^{\alpha},$$

where Π is a suitable subset of Δ. An analogous decomposition of $\mathbf{q}$ works when $\mathbf{q}$ is a CR-structure of codimension 1 and $\mathbf{g}$ is a reductive Lie-algebra of the first category as proved by Gigante and

Tomassini, [GT]. We exhibite a class of Levi-flat CR-structures on a reductive Lie-algebra which are not LCR.
Our investigation of CR-semisimple LCR-algebras concludes by proving that on any noncompact reductive Lie-algebra a semisimple LCR-structure exists. Moreover, the only reductive Lie-algebra without LCR-structure are the compact ones which have a one-dimensional centre (or which don't have centre), Theorem 2.2.3. The other compact ones are endowed with an abelian LCR-structure.

Finally, in the spirit of the classical root space decomposition of semisimple Lie-algebras, a decomposition theorem is given in terms of Cartan sub-LCR-algebras and CR-roots for CR-semisimple LCR-algebras (Theorem 4.3.1). An interesting consequence is that a CR-semisimple LCR-algebra $\mathbf{g}$ with LCR-structure $\mathbf{q}$ admits a real form $\mathbf{g}_0^*$ whose an ideal $\mathbf{p}^*$ is a compact real form of $\mathbf{q}$. This is the CR-analogous of the classical theorem: every complex semisimple Lie-algebra has a compact real form, [HE].

Chapter 1

CR-structures.

1.1 Introduction to Chapter 1.

This Chapter is devoted to the definition of main concepts about CR-structures on a Lie-algebra $\mathfrak{g}_0$. A CR-structure is a complex structure given on a subspace $\mathfrak{p}$ of $\mathfrak{g}_0$. So, the complex structures may be viewed as the CR-structures on the whole $\mathfrak{g}_0$. As CR-structures, they are Levi-flat; where the Levi-flatness will assume the meaning specified in the following Section. The study of CR-structures has a complex counterpart: each CR-structure may be read in the terms of a complex subalgebra $\mathfrak{q}$ of the complexified $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbf{R}} \mathbf{C}$, such that $\mathfrak{q} \cap \overline{\mathfrak{q}} = \{0\}$. Remark that the overlined objects are the conjugated ones, with respect of the conjugation τ induced by the complexification $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbf{R}} \mathbf{C}$. We shall often say that $\mathfrak{q}$ is a CR-structure on $\mathfrak{g}_0$. Via this complex subalgebra, we define two subclasses of the set of CR-structures $CR(\mathfrak{g}_0)$. The class $LfCR(\mathfrak{g}_0)$ whose elements are characterised by the fact that the subspace $\mathfrak{q} \oplus \overline{\mathfrak{q}}$ is a complex subalgebra. They are said *Levi-flat*. And the class $LCR(\mathfrak{g}_0)$ for which $\mathfrak{q}$ is a complex ideal. Of course, the following inclusions are given

$$CR(\mathbf{g}_0) \supseteq LfCR(\mathbf{g}_0) \supseteq LCR(\mathbf{g}_0).$$

The description of these particular classes will be the aim of Chapter 2.

A Lie-algebra $\mathbf{g}_0$ on which is given the CR-structure $(\mathbf{p}, J)$ is said to be a *CR-algebra.* In Section 1.3, we study and the subalgebras which admits a CR-structure induced by $(\mathbf{p}, J)$; and the Lie-homomorphisms with respect of which $\mathbf{p}$ is invariant and which commute with J. These subalgebras are said *sub-CR-algebras*, while the Lie-homomorphisms are the *CR-homomorphisms.* Notice that a sub-CR-algebra is a real subalgebra $\mathbf{h}_0$ of $\mathbf{g}_0$ on which $(\mathbf{p}, J)$ induces the CR-structure $(\mathbf{p} \cap \mathbf{h}_0, J_{\mathbf{p}\cap\mathbf{h}_0})$. For simplicity, we often say that a complex subalgebra $\mathbf{h}$ of the complexified $\mathbf{g}$ is a *sub-CR-algebra* when $\mathbf{h}$ is the complexified of a sub-CR-algebra $\mathbf{h}_0$. In particular, $\mathbf{g}$ *CR-algebra* means that $\mathbf{g}_0$ is a CR-algebra, in the sense that it is endowed with a CR-structure $(\mathbf{p}, J)$. In the terms of sub-CR-algebras, the concepts of CR-nilpotence, CR-solvability and CR-semisimplicity will be introduced in Chapter 3.

In Section 1.4, we consider the semidirect sums of two Lie-algebras. On them, we describe the CR-structures splitted in the "natural" way: i.e., the ones for which the underling subspace $\mathbf{p}$ is the sum of $\mathbf{p}_1$ and $\mathbf{p}_2$, which are subspaces of the two Lie-algebras. Furthermore, we construct some CR-structures even when the two factors do not admit CR-structures. The particular case of reductive Lie-algebras is studied.

On reductive Lie-algebras a family of Levi-flat CR-structure which are not Lie-s is exhibited.

In the Appendix, we give three examples of real Lie-algebras g_i, $i = 1, 2, 3$ which show that the inclusions of the CR-classes are proper. Precisely, we shall compute that

$$CR(\mathrm{g}_1) = Gr(2,3) \supset LfCR(\mathrm{g}_1) = \emptyset$$

$$\begin{aligned} CR(\mathrm{g}_2) &= Gr(2,3) \supset LfCR(\mathrm{g}_2) = \{L(X,Y) : Y^1 = (Y^2)^2 + (Y^3)^2, \\ &\quad (X^1)^2 + 1 = (X^2)^2 + (X^3)^2\} \supset LCR(\mathrm{g}_2) = \emptyset \end{aligned}$$

$$\begin{aligned} CR(\mathrm{g}_3) &= Gr(2,4) \supset LfCR(\mathrm{g}_3) = LCR(\mathrm{g}_3) = \\ &= \{\mathrm{p} \in Gr(2,4) : \mathrm{p} \text{ contains a fixed vector } E_4\}. \end{aligned}$$

1.2 Basic definitions.

Let g_0 be a real Lie algebra. In the sequel, g is its complexification $\mathrm{g}_0 \otimes_{\mathbf{R}} \mathbf{C}$. The conjugation with respect to g_0 is the real Lie-isomorphism τ. The conjugated element of X is also denoted as $\overline{X}$. Moreover, we shall write with $[,]$ and the real and the complex Lie bracket. Just by definition of real Lie-isomorphism it is $[\overline{X}, \overline{Z}] = \overline{[X, Z]}$, which is translated, in terms of adjoint transformations, as $ad_{\overline{Z}} = \tau ad_Z \tau$. Obviously, if $\mathfrak{a}$ is a complex subalgebra, $\overline{\mathfrak{a}}$ is a complex subalgebra, too. The object of this thesis may be seen as the complex subalgebras which do not intersect their conjugated ones.

Definition 1.2.1 *A CR-structure on* $\mathfrak{g}_0$ *is a pair* $(\mathfrak{p}, J)$ *composed by a linear subspace* $\mathfrak{p}$ *of* $\mathfrak{g}_0$ *and an endomorphism* $J : \mathfrak{p} \to \mathfrak{p}$ *such that*

1) $J^2 = -id$

2) $[X, Y] - [JX, JY] \in \mathfrak{p}, \forall X, Y \in \mathfrak{p}$

3) $[JX, JY] = [X, Y] + J[JX, Y] + J[X, JY], \forall X, Y \in \mathfrak{p}$.

In this case, $\mathfrak{g}_0$ *is said to be a CR-algebra.*

Lemma 1.2.2 *If* $(\mathfrak{p}, J)$ *is a CR-structure on* $\mathfrak{g}_0$, *then the complex subspace* $\mathfrak{q} \doteq \{X - iJX | X \in \mathfrak{p}\}$ *is a subalgebra of* $\mathfrak{g}$ *which does not intersect* $\overline{\mathfrak{q}}$.

Such a Lemma suggests a "complex" equivalent definition of a CR-structure which is more useful in view of the approach of this thesis,

Definition 1.2.3 *A CR-structure* $\mathfrak{q}$ *on* $\mathfrak{g}_0$ *is a complex subalgebra* $\mathfrak{q}$ *of* $\mathfrak{g}$, *such that* $\mathfrak{q} \cap \overline{\mathfrak{q}} = \{0\}$.

Proposition 1.2.4 *Given a CR-structure* $\mathfrak{q}$ *on* $\mathfrak{g}_0$, *there exist* r *real vectors* $X_i \in \mathfrak{g}_0$ *such that* $\mathfrak{g} = \mathfrak{q} \oplus \overline{\mathfrak{q}} \oplus \mathfrak{v}$, *where* $\mathfrak{v} = \oplus_{i=1}^{r} \mathbf{C} X_i$. *The complex vector space* $\mathfrak{v}$ *is* τ*-stable. The integer* $r = dim_{\mathbf{C}} \mathfrak{v}$ *is said the real codimension of* $\mathfrak{q}$. *Whenever* $r = 0$, $\mathfrak{q}$ *is a complex structure.*

Proof: any basis (X_i) which completes in $\mathbf{g}_0$ a basis of $\mathbf{p} = \Re\mathbf{q}$ satisfies the proposition. ■

The datum of a CR-structure $\mathbf{q}$ is equivalent to the pair $(\mathbf{p}, J)$ given in the Definition 1.2.1.

Lemma 1.2.5 *Let* $\mathbf{p}$ *be the real part of* $\mathbf{q}$, $Re\mathbf{q}$, *the CR-structure* $\mathbf{q}$ *determines a linear endomorphism* $J : \mathbf{p} \to \mathbf{p}$ *such that* $X - iJX$ *stays in* $\mathbf{q}$, *for any* $X \in \mathbf{p}$. *Moreover all the elements of* $\mathbf{q}$ *assumes the form* $X - iJX$.

Proof: the firs part is a trivial consequence of the fact that $\mathbf{q} \cap \overline{\mathbf{q}} = \{0\}$. Consider now $Z \in \mathbf{q}$; obviously ReZ stays in $\mathbf{p}$, and, consequently, $ReZ - iJReZ$ is in $\mathbf{q}$. The element $W_Z = Z - (ReZ - iJReZ)$ stays in $\mathbf{q}$. A trivial computation says that $W_Z = -\overline{W}_Z$, so W_Z vanishes and $ImZ = -JReZ$. ■

The above Lemma depends only on the fact that $\mathbf{q}$ is a linear subspace which does not intersect $\overline{\mathbf{q}}$. The fact that $\mathbf{q}$ is a subalgebra links J and the real Lie-product $[,]$.

Lemma 1.2.6 *The endomorphism* J *verifies the conditions*

1) $J^2 = -id$

2) $[X,Y] - [JX, JY] \in \mathbf{p}, \forall X, Y \in \mathbf{p}$

3) $[JX, JY] = [X,Y] + J[JX,Y] + J[X,JY], \forall X, Y \in \mathbf{p}$.

This means that J *is a integrable complex structure on* $\mathbf{p}$.

Thus, we have completely proved the equivalence between the real and the complex definition. In the following, we shall denote both with

(p, J) and with q the CR-structure. In each context the notation will be evident.

A particular interest is taken by those CR-structures which have more algebraic structure. In the sense that p is either a subalgebra or an ideal.

Definition 1.2.7 *A CR-structure* q *is said to be Levi-flat if* $\tilde{\mathrm{q}} \doteq \mathrm{q} \oplus \overline{\mathrm{q}}$ *is a complex subalgebra. When* q *is a complex ideal,* q *is said a Lie-CR-structure, or a LCR-structure. In the first case* g_0 *and* g *are said Levi-flat CR-algebras. In the last, LCR-algebras.*

The following examples prove that there are CR-structures which are not Levi-flat; and Levi-flat ones which are not LCR-structures. Some example of the existence of each kind of CR-structures are given in the Appendix.

Example 1 *Let us consider the complex three-dimensional linear space* $\mathbf{C}^3$. *Let* X_1, X_2 *be two vectors such that* $\tau X_1 \neq \pm X_1$, $X_2 = -\tau X_2$ *and let* $(X_1, \tau X_1, X_2)$ *be a basis of* $\mathbf{C}^3$. *If we define*

$$[X_1, X_2] = 0$$
$$[X_1, \tau X_1] = X_2$$

$\mathrm{g} = (\mathbf{C}^3, [,])$ *is a solvable Lie-algebra. Taken* $\mathrm{q}_1 \doteq \mathbf{C}X_1$, *we have that* $\mathrm{q}_1 \cap \overline{\mathrm{q}}_1 = \{0\}$ *and* $[\mathrm{q}_1, \overline{\mathrm{q}}_1] = \mathbf{C}X_2$. *So,* q_1 *is a CR-structure which is not Levi-flat.*

Example 2 *Let* g_0 *be a real semisimple Lie-algebra and* h_0 *be an its Cartan subalgebra. Then,* g *and* h *are their complexifications. Since* h *is abelian, any nonvanishing subspace* q *of* h *such that* $q \cap \overline{q} = \{0\}$ *defines a Levi-flat CR-structure on* g_0 *and a LCR-structure on* h_0. *Moreover* q *can not be an ideal of* g. *So it is not a LCR-structure.*

Let us conclude this Section with two results about the algebraic properties of p. Thus, we give the "real" definitions of Levi-flat and Lie's CR-structure. In the sequel we denote with $u = \oplus_{j=1}^{r} \mathbf{R} X_j$ and p the real part of v and q, respectively. We shall write $\tilde{q}$ for the direct sum $q \oplus \overline{q}$. As we have already remarked (Proposition 1.2.4), we have the decompositions $g_0 = p \oplus u$ and $g = \tilde{q} \oplus v$.

Proposition 1.2.8 *The linear subspace* p *is a real subalgebra if and only if* $\tilde{q}$ *is a complex one. This means that a CR-structure is Levi-flat if and only if* p *is a real subalgebra.*

Let us give the proof. In particular, we shall show that $[p, p]$ is included in p if and only if $[q, \overline{q}]$ is contained in $q \oplus \overline{q}$. If p is a subalgebra, consider X, Y in p, and the elements

$$[X - iJX, Y + iJY] = [X, Y] + [JX, JY] + i([X, JY] - [JX, Y])$$

$$2Z \doteq [X, Y] + [JX, JY] + J([X, JY] - [JX, Y])$$

$$2W \doteq [X,Y] + [JX,JY] - J([X,JY] - [JX,Y]).$$

Trivially it is $Z, W \in \mathrm{p}$ and $[X - iJX, Y + iJY] = Z + W + iJ(W - Z) \in \mathrm{q} \oplus \overline{\mathrm{q}}$.

Vice versa if there are $Z, W \in \mathrm{p}$ such that $[X - iJX, Y + iJY] = Z + iJW$, then $[X,Y] + [JX,JY] = Z \in \mathrm{p}$. Since, by definition, $[X,Y] - [JX,JY] \in \mathrm{p}$, it follows that $[X,Y]$ is in p. ∎

An analogous result follows about LCR-structures.

Proposition 1.2.9 *A CR-structure* q *is a LCR-structure if and only if* p *is a real ideal and J is ad_X-invariant. Obviously, a LCR-structure is Levi-flat.*

Remark 1.2.10 *Of course, even in this case, the more geometrical definitions are those given in the real terms. That is, the CR-structure* (p, J) *is Levi-flat, whenever* p *is a real subalgebra; it is a LCR-structure, whenever* p *is a real ideal and J is invariant under all the adjoint derivations ad_X. The complex definitions have been introduced, since they have an easier application in the direct computations.*

1.3 Sub-CR-algebras.

In the family of all the real subalgebras $\mathbf{h}_0$, we are interested in those on which (p, J) induces a CR-structure. Let $\mathbf{h}$ denote the complexification of h_0. In the general case, the subalgebras $\mathrm{h} \cap \mathrm{q}$ and $\mathrm{h} \cap \overline{\mathrm{q}}$ are not conjugated. Moreover, they may have not the same dimension. So we give the following

Definition 1.3.1 *The complex subalgebra* h *is a sub-CR-algebra if it is* τ*-stable and it admits the CR-structure* $\mathrm{h} \cap \mathrm{q}$ *induced by* q. *When* $\mathrm{h} \cap \mathrm{q}$ *is a Levi-flat CR-structure,* h *is said a Levi-flat sub-CR-algebra. When* $\mathrm{h} \cap \mathrm{q}$ *is a LCR-structure,* h *is said a Lie-sub-CR-algebra. Let* h *be an ideal. Then we speak, respectively, of a CR-ideal, a Levi-flat CR-ideal and a CR-ideal of Lie. Moreover, in the case that* q *is a LCR-structure,* h *is said a sub-LCR-algebra or a LCR-ideal. When* h *is a sub-CR-algebra and* $\mathrm{h} \cap \mathrm{q}$ *vanishes,* h *is said trivial. If* $\mathcal{D}\mathrm{h} \cap \mathrm{q}$ *vanishes.* h *is said CR-abelian.*

Example 3 *Let* g *be the Lie-algebra of real* $2n \times 2n$*-matrices,* $\mathrm{gl}(2n)$ *and* p *be the subspace of diagonal ones. When* A *is in* p*, define the CR-structure* J *as*

$$(JA)_i = -A_{n+i} \text{ and } (JA)_{n+i} = A_i, \text{ where } i \leq n.$$

Consider, now, the ideal $\mathrm{sl}(2n)$ *whose elements have trace vanishing. Such an ideal is not a CR-ideal. In fact, there are elements of* $\mathrm{p} \cap \mathrm{sl}(2n)$ *whose image via* J *has not null trace:* $J\begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} = I_{2n}$. *Examples of sub-CR-algebra are provided by the space of upper triangular matrices and by* $\mathrm{sl}(n) \oplus \mathrm{sl}(n)$.

Proposition 1.3.2 *The subalgebra* h *is a nontrivial sub-CR-algebra if and only if* $\tau(\mathrm{h}\cap \mathrm{q}) = \mathrm{h}\cap\overline{\mathrm{q}} \neq \{0\}$. *The same result is true in all the other cases.*

Of course, the complex definition 1.3.1 means that $(\mathrm{h}_0\cap \mathrm{p}, J_{\mathrm{h}_0\cap\mathrm{p}})$ is a CR-structure on h_0. The equivalence between these facts is given by the

Proposition 1.3.3 *The restriction of* J *to* $\mathrm{h}_0\cap\mathrm{p}$ *is an integrable complex structure. Vice versa, if* J *is an integrable complex structure on* $\mathrm{h}_0\cap\mathrm{p}$, $\mathrm{h}\cap\mathrm{q}$ *is a sub-CR-algebra.*

Corollary 1.3.4 *The intersection* $\mathrm{h}\cap\mathrm{q}$ *vanishes if and only if* $\mathrm{h}_0\cap\mathrm{p}$ *vanishes.*

Proof: the above Proposition may be written as

$$\mathrm{h}\cap\mathrm{q}=\{0\}\Leftrightarrow\begin{cases}\mathrm{h}_0\cap\mathrm{p}=\{0\} & \\ \mathrm{h}_0\cap\mathrm{p}\neq\{0\} & J \text{ does not map } \mathrm{h}_0\cap\mathrm{p} \text{ in itself}\end{cases}$$

Let us prove that the second case can not occur. Take the subalgebra $\mathrm{h}_0' \doteq \mathrm{h}_0\cap\mathrm{p}+J(\mathrm{h}_0\cap\mathrm{p})$. Then h_0' is invariant under J and intersects p. Thus, its complexified h' intersects q and it is contained in h: a contradiction. ∎

Hence, the sub-CR-algebras h_0 are characterised by the condition

$$J(\mathrm{h}_0\cap\mathrm{p})\subseteq\mathrm{h}_0\cap\mathrm{p}\neq\{0\}.$$

Let us return to the complex situation. Since τ is a real Lie-isomorphism, when h is τ-stable its derived and its central series are composed by τ-stable elements. Moreover, there is the

Proposition 1.3.5 *Let* h *be a sub-CR-algebra. Then either* h *is CR-abelian or* $\mathcal{D}\mathrm{h}$ *is a sub-CR-algebra.*

Proof: $\tau(\mathcal{D}\mathrm{h} \cap \mathrm{q}) = \mathcal{D}\overline{\mathrm{h}} \cap \overline{\mathrm{q}} = \mathcal{D}\mathrm{h} \cap \overline{\mathrm{q}}$. A similar result is true even for $\mathcal{D}^k\mathrm{h}$ and $\mathcal{C}^k\mathrm{h}$. ■

Theorem 1.3.6 *Let* h *be a CR-ideal of* g *which does not contain* q, *then* $\mathrm{q}/\mathrm{h} \cap \mathrm{q}$ *is a CR-structure of* g/h. *Hence* g/h *is a CR-algebra, said the CR-quotient.*

Proof: since $\mathrm{q}\cap\mathrm{h}$ is an ideal of q, $\mathrm{q}/\mathrm{q}\cap\mathrm{h}$ is a Lie-subalgebra of g/h. On g/h consider the conjugation τ defined as $\tau[X] = \overline{X} + \overline{\mathrm{h}} = \overline{X} + \mathrm{h} = [\overline{X}]$. Take a real element $[Q] = \overline{[Q]}$ of $(\mathrm{q}/\mathrm{q}\cap\mathrm{h})\cap(\overline{\mathrm{q}}/\overline{\mathrm{q}}\cap\mathrm{h})$. By definition, there is $H \in \mathrm{h}$ such that $Q + H = \overline{Q}$: then it is $Q - \overline{Q} \in \mathrm{h} \cap (\mathrm{q} \oplus \overline{\mathrm{q}})$. Since, $\mathrm{h} = \mathrm{h}\cap\mathrm{q} \oplus \mathrm{h}\cap\overline{\mathrm{q}} \oplus \mathrm{h}_1$, $\mathrm{h}\cap\mathrm{q} \oplus \mathrm{h}\cap\overline{\mathrm{q}} = \mathrm{h}\cap(\mathrm{q}\oplus\overline{\mathrm{q}})$. So, $Q \in \mathrm{h}\cap\mathrm{q}$, and hence $[Q]$ vanishes. ■

The Lie-homomorphisms which send a CR-structure in another one, are said *CR-homomorphisms.* More precisely,

Definition 1.3.7 *Consider two CR-algebras* g *and* g'. *A Lie-homomorphism (resp. a derivation)* $\alpha : \mathrm{g} \to \mathrm{g}'$ *is said a CR-homomorphism (resp. a CR-derivation) if* α *intertwines* τ *and* τ' *and it maps* q *in* q'. *The set of all the CR-homomorphisms is denoted with* $Hom^*(\mathrm{g}, \mathrm{g}')$.

The restriction of α to the linear subspace p defines an homomorphism $\alpha : \mathrm{p} \to \mathrm{p}'$ which intertwines J and J'. Vice versa, an homomorphism $\alpha : \mathrm{g}_0 \to \mathrm{g}_0'$ which maps p into p' and intertwines J and J', defines a CR-homomorphism.

Example 4 *Let us return to Example 3, consider the matrix* e_{ij} *whose entries are* $\delta_{ik}\delta_{jh}$*, which has* 1 *in position* (i,j) *and* 0 *elsewhere. Define the real subspaces* $E_1 = \oplus_{i\leq n}\mathbf{R}e_{ii}$ *and* $E_2 = \oplus_{i\geq n}\mathbf{R}e_{ii}$*. The CR-homomorphisms are the Lie-homomorphisms which let both* E_1 *and* E_2 *invariant.*

Proposition 1.3.8 *Let* α *be an element of* $Hom^*(\mathrm{g},\mathrm{g}')$*, then* $Im\alpha$ *is a sub-CR-algebra of* g' *and* $\ker\alpha$ *is a CR-ideal of* g *(when* $\alpha|_{\mathrm{q}}$ *is not invertible). Moreover,* $\alpha\mathrm{q}$ *is a CR-structure of* $\alpha\mathrm{g}$*.*

Whenever, α is an isomorphism, then the two CR-algebras are said to be *CR-isomorphic* and the corresponding CR-structures are said to be *equivalent.*

1.4 Semidirect sums of CR-structures.

Take two Lie-algebras g_0 and g_0', and consider the CR-structures (p,J) on g_0 and (p',J') on g_0'. If δ is a Lie-homomorphism between g_0 and $Der(\mathrm{g}_0')$, a classical construction gives the *semidirect sum* of g_0' and g_0 by δ. Since the direct sum $\mathrm{g}_0'\oplus_\delta \mathrm{g}_0$ is defined on the linear space $\mathrm{g}_0'\oplus \mathrm{g}_0$, we would like to know when the pair $(\mathrm{p}_\oplus = \mathrm{p}'\oplus\mathrm{p}, J_\oplus = J'\oplus J)$ is a CR-structure, too. In this case, it is called *the semidirect sum* of the CR-structures (p,J) and (p,J'). A direct computation proves the

Proposition 1.4.1 *The pair* $(\mathrm{p}_\oplus, J_\oplus)$ *is a CR-structure on* $\mathrm{g}_0'\oplus_\delta \mathrm{g}_0$ *if and only if* $D_J(X) \doteq \delta(JX)+\delta(X)J'$ *is a CR-linear map, for all* X *in* p.

Corollary 1.4.2 *When* (p, J) *is a CR-structure on* g_0, $(\{0\} \oplus_\delta \mathrm{p}, J)$ *is a CR-structure, for all* g'_0 *and for all* δ.

Remind that when g'_0 is semisimple, any derivation is inner. So every $\delta : \mathrm{g}_0 \to Der(\mathrm{g}'_0)$ takes the form δ_B, for a suitable $B \in Hom(\mathrm{g}_0, \mathrm{g}'_0)$. If g_0 and g'_0 are endowed with CR-structures and B is a CR-homomorphism, the corresponding semidirect sum supports as CR-structure the semidirect sum of the two CR-structures.

Proposition 1.4.3 *Let* g'_0 *be a semisimple Lie-algebra, then the pair* $(\mathrm{p}_\oplus, J_\oplus)$ *is a CR-structure of any* $\mathrm{g}'_0 \oplus_{\delta_B} \mathrm{g}_0$, *with* $B \in Hom^*(\mathrm{g}_0, \mathrm{g}'_0)$: *where* $\delta_B(X) \doteq ad_{BX}$.

In the general case, notice that $(\mathrm{p}_\oplus, J_\oplus)$ is a Levi-flat CR-structure if and only if $[(U, X), (V, Y)] + [J(U, X), J((V, Y)]$ is in $\mathrm{p}_\oplus$, with U, V in p' and X, Y in p. This fact implies that (p', J') and (p, J) have to be Levi-flat CR-structures and that $\delta(X) + \delta(JX)J' \in \mathrm{gl}^*(\mathrm{p})$.

By Proposition 1.4.1, $D_J(JX) = \delta(JX)J' - \delta(X)$ is an element of $\mathrm{gl}^*(\mathrm{p}')$. So the further condition implies that the homomorphism δ maps p in $\mathrm{gl}^*(\mathrm{p}')$. Let us summarise the result in the following

Proposition 1.4.4 *The pair* $(\mathrm{p}_\oplus, J_\oplus)$ *is a Levi-flat CR-structure on* $\mathrm{g}'_0 \oplus_\delta \mathrm{g}_0$ *if and only if*

1. (p', J') *is a Levi-flat CR-structure on* g'_0;
2. (p, J) *is a Levi-flat CR-structure of* g_0;
3. $\delta(\mathrm{p}) \subseteq \mathrm{gl}^*(\mathrm{p}')$. ■

Via an analogous computation, it is possible to prove the

Proposition 1.4.5 *The pair* $(\mathrm{p}_\oplus, J_\oplus)$ *is a LCR-structure if and only if*

1. (p', J') *is a LCR-structure on* g_0'*;*
2. (p, J) *is a LCR-structure on* g_0*;*
3. $\delta(JX) = J'\delta(X), \forall X \in \mathrm{p}$*;*
4. $\delta(X)J' = J'\delta(X), \forall X \in \mathrm{g}_0$*;*
5. $\delta(X)\mathrm{p}' \subseteq \mathrm{p}', \forall X \in \mathrm{g}_0$*;*
6. $\delta(X)\mathrm{g}_0' \subseteq \mathrm{p}', \forall X \in \mathrm{p}$.

Let us consider a different case involving semidirect sums. Suppose that nor g_0 neither g_0' supports a CR-structure. Even in this case, it is possible that $\mathrm{g}_0' \oplus_\delta \mathrm{g}_0$ is endowed with a CR-structure. In fact, consider a subalgebra p in g_0 and an abelian one p' in g_0'. Let $E : \mathrm{p} \to \mathrm{p}'$ be a linear isomorphism such that $E[X, Y] = \delta(X)EY - \delta(Y)EX$, for all $X, Y \in \mathrm{p}$. Then, the pair $(\mathrm{p}_\oplus = \mathrm{p}' \oplus \mathrm{p}, J_E = \begin{pmatrix} 0 & E \\ -E^{-1} & 0 \end{pmatrix})$ is a CR-structure. The further condition $\delta(V)EX - \delta(Y)EU \in \mathrm{p}'$ characterises the Levi-flat CR-structures (p, J_E). Finally, when p' and p are ideals and $\delta(G)EX - \delta(Y)EH \in \mathrm{p}'$, $(\mathrm{p}_\oplus, J_E)$ is a LCR-structure.

If we focus our mind on LCR-structures, Proposition 1.4.5 assures that, if g_0' is endowed with a complex structure and if $\delta(X)$ is holomorphic, $\mathrm{g}_0' \oplus_\delta \mathrm{g}_0$ supports a LCR-structure, where g_0 is a generic real Lie-algebra. That will be the case of noncompact semisimple Lie-algebras where g_0 is the sum of the real factors and g_0' is the sum of the Cartan-classified ones, cf. Chapter 2, Section 2. Another example is given by a reductive Lie-algebra. In fact, in that case the algebra is the direct sum

of its centre and of a semisimple Lie-subalgebra. So, a LCR-structure is direct sum of an abelian LCR-structure with a semisimple one. Such a situation is a particular case of Levi-Mal'cev decomposition. Such a decomposition will be the object of the following Chapter.
Let us describe the particular case of a reductive Lie-algebra g_0. Such an algebra is given by the direct sum of its centre and of its derived (which is semisimple): $\mathrm{g}_0 = \zeta(\mathrm{g}_0) \odot \mathcal{D}\mathrm{g}_0$.

In the following, such a decomposition will take a central position. In fact, we look only for the CR-structures splitted as $(\mathrm{p} = \mathrm{p}_\mathrm{a} \oplus \mathrm{p}_\mathrm{s} J = \begin{pmatrix} J_\mathrm{a} & E \\ F & J_\mathrm{s} \end{pmatrix})$. This choice is, in general, restrictive. While, if we consider just the LCR-structures. it is not. In fact, let p be an ideal of g_0. Hence, $\mathrm{p}_\mathrm{a} = \mathrm{p} \cap \zeta(\mathrm{g}_0)$ is its radical. Take an its Levi-subalgebra p_s. Since p_s is a semisimple subalgebra, it is included in the Levi-subalgebra $\mathcal{D}\mathrm{g}_0$. Thus, p takes the desired form.

Now, take a subspace $\mathrm{p} = \mathrm{p}_\mathrm{a} \oplus \mathrm{p}_\mathrm{s}$. Then, impose that $J = \begin{pmatrix} J_\mathrm{a} & E \\ F & J_\mathrm{s} \end{pmatrix}$ is an integrable complex structure on it. By definition, the following relations have to be satisfied

By a direct computation, it is possible to show that the following relations have to be verified:

1. $J_\mathrm{a}^2 + EF = -id_{\mathrm{p}_\mathrm{a}}$
2. $J_\mathrm{s}^2 + FE = -id_{\mathrm{p}_\mathrm{s}}$
3. $J_\mathrm{a}E + EJ_\mathrm{s} = 0$
4. $J_\mathrm{s}F + FJ_\mathrm{a} = 0$
5. $[ImF, ImF] = 0$
6. $[X, Y] - [J_\mathrm{s}X, J_\mathrm{s}Y] \in \mathrm{p}_\mathrm{s}$

7. $[J_sX, J_sY] = [X, Y] + J_s[J_sX, Y] + J_s[X, J_sY]$

8. $[ImF, p_s] \in KerE$

9. $E[J_sX, J_sY] = E[X, Y]$

10. $ad_{FA}J_s = J_s ad_{FA}$.

Corollary 1.4.6 *Any reductive Lie-algebra is endowed with a CR-- structure.*

Proof: consider, in fact, an abelian subalgebra p_s, whose dimension is less or equal to $\dim\zeta(g_0)$ (such a subalgebra exists. In fact, any linear subspace of the Cartan subalgebra h of s is abelian); and a linear monomorphism $E : p_s \to \zeta(g_0)$. Then, the pair $(p = Ep_s \oplus p_s, J_E = \begin{pmatrix} 0 & E \\ -E^{-1} & 0 \end{pmatrix})$ is a CR-structure on g_0. In particular, since p is abelian, (p, J_E) is Levi-flat. Obviously, (p, J_E) can not be a Lie's one, otherwise p_s would be an abelian ideal of s. Such a construction provides a "large" family of Levi-flat CR-structures which are not Lie's. ■

The ten relations provide other interesting families of splitted CR-structures on a reductive Lie-algebra. Suppose that (p_a, J_a) and (p_s, J_s) are CR-structures on $\zeta(g_0)$ and $\mathcal{D}g_0$, respectively. Then

i) the direct sum $(p = p_a \oplus p_s, J_a \oplus J_s)$ is a CR-structure on g_0;

ii) whenever $E : p_s \to p_a$ satisfies

$$J_aE + EJ_s = 0$$

$$E[J_sX, J_sY] = E[X, Y],$$

the pair $(p, J = \begin{pmatrix} J_a & E \\ 0 & J_s \end{pmatrix})$ defines a CR-structure on g_0;

iii) whenever $F : \mathrm{p_a} \to \mathrm{p_s}$ satisfies $J_sF + FJ_a = 0$ and $ad_{FX}J_s = J_s ad_{FX}$, $\forall X \in \mathrm{p_a}$, $(\mathrm{p}, J = \begin{pmatrix} J_a & 0 \\ F & J_s \end{pmatrix}$ is a CR-structure.

In Chapter 2, we shall show that the only LCR-structures of a reductive Lie-algebra take the form $(\mathrm{p}_\oplus = \mathrm{p_a} \oplus \mathrm{p_s}, J_\oplus = J_a \oplus J_s)$.

In conclusion, let us observe that even the Levi-flat CR-structures are given on splitted spaces.

Proposition 1.4.7 *A real subalgebra* p *of a reductive Lie-algebra* $\mathrm{g_0}$ *is reductive.*

Proof: remind that a Lie-algebra is reductive if and only if its adjoint representation is semisimple. Then, take X in p and an ad_X-invariant subspace V of p. Since $\mathrm{g_0}$ is reductive, there exists an ad_X-invariant subspace W of $\mathrm{g_0}$ such that $\mathrm{g_0} = V \oplus W$. Let π_W be the projection on W defined by the given decomposition. Since V is included in p, $\pi_W\mathrm{p}$ is contained in p and it coincides with $\mathrm{p}\cap W$. Obviously, $\mathrm{p} = V \oplus \mathrm{p} \cap W$ and $\mathrm{p} \cap W$ is invariant under ad_X. ■

Corollary 1.4.8 *Whenever* p *is a subalgebra,* p *is decomposed as* $\mathrm{p} = \zeta(\mathrm{p}) \odot \mathcal{D}\mathrm{p}$. *Notice that* $\mathcal{D}\mathrm{p}$ *is included in* $\mathcal{D}\mathrm{g_0}$ *while* $\zeta(\mathrm{p})$ *is not necessary in* $\zeta(\mathrm{g_0})$. ■

In any case, a Levi-flat CR-structure satisfies the above ten equations.

1.5 Appendix.

We study three examples of Lie-algebras of low dimension. On each of them, all the CR-structures are studied. They are interesting because they furnish examples of CR-structures which are not Levi-flat; and of Levi-flat CR-structures which are not Lie's.

Example 5 *Let S^3 be the three-dimensional sphere. It is a compact Lie-group, whose Lie-algebra is* $\mathrm{su}(2) = \{A \in \mathrm{gl}(2, \mathrm{C}) : trA = 0, A^t + \overline{A} = 0\}$*. The generic element of* $\mathrm{su}(2)$ *is* $\begin{pmatrix} ix & u+iv \\ -u+iv & -ix \end{pmatrix}$*. Hence, a basis is given by* $E_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$*. Furthermore, the Lie-product is defined by*

$$[E_1, E_2] = -2E_3$$

$$[E_1, E_3] = 2E_2$$

$$[E_2, E_3] = -2E_1.$$

First of all, remark that the centre of $\mathrm{su}(2)$ *vanishes. Hence, since it is compact, it is simple. Then,* $\mathrm{su}(2)$ *has no ideals, and, hence, no LCR-structures.*

Remind, now, that a CR-structure is given on an even-dimensional subspace p*. So, we study the planes* $\mathrm{p} \subseteq \mathrm{su}(2)$*. In the case that* p *is a subalgebra, or it is abelian either it is solvable. Since the product of two vectors is given by*

$$[X, Y] = 2(X^3Y^2 - X^2Y^3)E_1 + 2(X^1Y^3 - X^3Y^1)E_2 + 2(X^2Y^1 - X^1Y^2)E_3,$$

it vanishes if and only if they are linearly dependents. This means that there are no abelian planes.

Consider now a solvable bidimensional subalgebra p. *It is possible to find two vectors* $X, Y \in \mathrm{p}$ *such that*

1. $\mathrm{p} = \mathbf{R}X \oplus \mathbf{R}Y$
2. $[X, Y] = Y$.

The second relation implies that

$$(Y^2)^2 + (Y^3)^2 = -(Y^1)^2,$$

where the Y^i*'s are the components of* Y *with respect of* E_i. *Obviously, the only solution is* $Y = 0$. *Hence, there are nor bidimensional subalgebras, neither Levi-flat CR-structures. Otherwise, any plane* $\mathrm{p} = \mathbf{R}X \oplus \mathbf{R}Y$ *admits the complex structure* $JX \doteq Y$, $JY \doteq -X$.

In conclusion, the Lie-algebra su(2) has no bidimensional subalgebras. Thus, the sphere S^3 does not admit Levi-flat CR-structure.

Example 6 *Consider the matrices* $E_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, *and the space* $\mathrm{g}_0 = \oplus_i \mathbf{R} E_i$. *Since,*

$$[E_1, E_2] = -2E_3$$

$$[E_1, E_3] = 2E_2$$

$$[E_2, E_3] = 2E_1,$$

$\mathfrak{g}_0$ *is a real Lie-algebra, whose centre vanishes. Let us write the Lie-product of two vectors* X *and* Y

$$[X,Y] = 2(X^2Y^3-X^3Y^2)E_1+2(X^1Y^3-X^3Y^1)E_2+2(X^2Y^1-X^1Y^2)E_3.$$

The following system defines the eigenvectors of ad_X*:*

$$\begin{cases} X^2Y^3 - X^3Y^2 = \lambda Y^1 \\ X^1Y^3 - X^3Y^1 = \lambda Y^2 \\ X^2Y^1 - X^1Y^2 = \lambda Y^3 \end{cases}$$

Since one of the Y^i*'s does not vanishes, let us pose* $Y^1 = 1$. *Then, the system becomes*

$$\begin{cases} Y^3X^2 - Y^2X^3 = \lambda \\ X^3 = Y^3X^1 - \lambda Y^2 \\ X^2 = X^1Y^2 + \lambda Y^3 \end{cases}$$

so $Y^2 = \cos\alpha$, $Y^3 = \sin\alpha$, $Y = (1, \cos\alpha, \sin\alpha)$

Let us write the second and the third equations as

$$\begin{cases} Y^2 = \frac{X^1X^2-\lambda X^3}{(X^1)^2+\lambda^2} \\ Y^3 = \frac{X^1X^3+\lambda X^2}{(X^1)^2+\lambda^2} \end{cases}$$

this means that, when λ *is a nonvanishing eigenvalue,* λ *is a zero of*

$$(X^1X^3 + \lambda X^2)^2 + (X^1X^2 - \lambda X^3)^2 = ((X^1)^2 + \lambda^2)^2,$$

and then of

$$\lambda^2 = (X^2)^2 + (X^3)^2 - (X^1)^2.$$

So, $tr(ad_X)$ *vanishes, for all* $X \in \mathfrak{g}_0$*, and* $\mathfrak{g}_0$ *is said unimodular. A classical result about unimodular three-dimensional algebras says that* *the Killing form is given by* $B(X,Y) = -8(X^1Y^1 - X^2Y^2 - X^3Y^3)$, *cf. the Appendix to Chapter 2. Hence,* $\mathfrak{g}_0$ *is simple. In particular, it does not admit LCR-structures and it is isomorphic to* $\mathrm{sl}(2,\mathbf{R})$.

Since a CR-structure of $\mathfrak{g}_0$ *is supported by a plane, let us study the planes and the bidimensional subalgebras. When* $\mathfrak{p} = \mathbf{R}X \oplus \mathbf{R}Y$ *is a subalgebra,* $\mathfrak{p}$ *has to be solvable. In fact,* X *and* Y *commutes if and only if they are linear dependents. Let us consider* X *and* Y *in* $\mathfrak{p}$ *such that* $[X,Y] = Y$. *Imposing this condition, we obtain the linearly independents vectors*

$$Y_\alpha = (1, \cos\alpha, \sin\alpha)$$

$$X_{a,\alpha} = (a, \sin\alpha + a\cos\alpha, a\sin\alpha - \cos\alpha).$$

Then, $\forall a,\alpha \in \mathbf{R}$, $\mathfrak{p}_{a,\alpha} = \mathbf{R}Y_\alpha \oplus \mathbf{R}X_{a,\alpha}$ *is a solvable subalgebra. Its endomorphism* $J_{a,\alpha}$, *which sends* Y_α *in* $X_{a,\alpha}$ *and* $X_{a,\alpha}$ *in* Y_α, *defines a Levi-flat CR-structure on* $\mathfrak{g}_0$.
Remind that $\mathfrak{p}_{a,\alpha}$ *does not depend on* a. *In fact, we may write* $X_{b,\alpha}$ *as* $X_{a,\alpha} + (b-a)Y_\alpha$.

Finally, observe that the generic CR-structures are more than the Levi-flat ones. In fact, the vectors Y_α *belong to the cone* Γ *of equation* $X^1 = (X^2)^2 + (X^3)^2$, *while the vectors* $X_{a,\alpha}$ *are on the hyperboloid* H *of equation* $(X^1)^2 + 1 = (X^2)^2 + (X^3)^2$. *A plane, which does not intersect the above cone, supports a CR-structure but it is not a subalgebra.*

So, $\mathfrak{g}_0$ has no LCR-structure. Any its plane defines a CR-structure. While the Levi-flat ones are generated by a suitable pair of vectors taken in Γ and in H.

Example 7 *Consider the real linear space* $\mathfrak{g}_0$ *of complex matrices*

$$\begin{pmatrix} 0 & z & w \\ 0 & 0 & \bar{z} \\ 0 & 0 & 0 \end{pmatrix},$$

and the matrices e_{ij} *which have* 1 *in the position* (i,j) *and* 0 *elsewhere. A basis of* $\mathfrak{g}_0$ *is given by* $E_1 = e_{12} + e_{23}$, $E_2 = i(e_{12} - e_{23})$, $E_3 = e_{13}$, $E_4 = ie_{13}$. *A trivial computation shows that the only noncommuting matrices are* E_1 *and* E_2, *whose product is*

$$[E_1, E_2] = -2E_4.$$

Hence, $\mathcal{D}\mathfrak{g}_0 = \mathbf{R}E_4$ *and* $\mathcal{D}^2\mathfrak{g}_0 = 0$. *So,* $\mathfrak{g}_0$ *is a solvable Lie-algebra. By definition, the vector* E_4 *stays in all the subalgebras with dimension greater than* 2. *Moreover,* $\mathfrak{p}_X \doteq \mathbf{R}X \oplus \mathbf{R}E_4$ *is the generic bidimensional ideal. So, we may conclude that the Levi-flat CR-structures of* $\mathfrak{g}_0$ *are LCR-structures and are given by* $(\mathfrak{p}_X, J_X)$, *where* $J_X X = E_4$ *and* $J_X E_4 = -X$.

In conclusion, the Levi-flat CR-structures are defined by the planes containing E_4. Each of them is a LCR-structure.

Chapter 2

LCR-structures.

2.1 Introduction to Chapter 2.

In [SN], the author studies the left-invariant complex structures on reductive Lie-algebras. He considers a real reductive Lie-algebra $\mathfrak{g}_0$ endowed with an invariant complex structure. Hence, the complexification of $\mathfrak{g}_0$, $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbf{R}} \mathbf{C}$, may be decomposed as $\mathfrak{g} = \mathfrak{q} \oplus \overline{\mathfrak{q}}$, where $\mathfrak{q}$ is a complex subalgebra. Snow studies the regular complex structures. where regular means that there exists a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{h} = \overline{\mathfrak{h}}$ and $[\mathfrak{h}, \mathfrak{q}] \subseteq \mathfrak{q}$.
A regular $\mathfrak{q}$ can be written as

$$\mathfrak{q} = \mathfrak{q} \cap \mathfrak{h} \oplus \oplus_{\alpha \in \Pi} \mathfrak{g}^{\alpha},$$

where Π is a suitable subset of the root set Δ. Finally, Snow shows that every complex structure is regular, when it is given on a reductive Lie-algebra of the first category. Remind that in these algebras the involution determined by a Cartan decomposition is an inner automorphism. Such results have been translated by [GT] in terms of

CR-structures on reductive Lie-algebras of the first category: the authors study the case of real codimension 1. With this further hypothesis, they prove that there exists a compact Cartan subalgebra $\mathbf{h}_0$ of $\mathbf{g}_0$ on which the CR-structure $\mathbf{q}$ induces a CR-structure. Moreover, they find a subset $\Delta^+ \subseteq \Delta$ which determines a decomposition similar to the Snow's one. Two cases are possible: either $\mathbf{q} = \mathbf{q} \cap \mathbf{h} \oplus \oplus_{\alpha>0}\mathbf{g}^\alpha$, or $\mathbf{q} = \mathbf{q} \cap \mathbf{h} \oplus \oplus_{\alpha>0,\alpha\neq\mu}\mathbf{g}^\alpha \oplus \mathbf{C}(H + X^\mu)$, where $H = \overline{H} \in \mathbf{h}$. In this Chapter we explore and classify all the LCR-structures on a Lie-algebra. With respect of [GT] we study a case in which the Lie-algebra is more generic (in fact, it has not to be reductive of the first category), while the CR-structure is more particular, since it is a Lie's one. Moreover, our approach does not use Cartan subalgebras and their corresponding root spaces. Chapter 4 will be devoted to this point of view. In the present Chapter, we consider the Levi-Mal'cev decomposition. Thus, we have to study LCR-structures in the semisimple and in the solvable cases (Sections 2.2 and 2.3): in the first one the LCR-structures are sums (in the sense of Proposition 2.2.5) of simple ideals endowed with a complex structure (described by Cartan in the classical classification, [HE]); in the second one they are given on even-dimensional ideals $\mathbf{p}$, decomposed as $\mathbf{p} = \mathbf{u} \oplus A\mathbf{u}$, by the endomorphism $J_A \doteq \begin{pmatrix} 0 & A \\ -A^{-1} & 0 \end{pmatrix}$. Finally, Section 2.4 concludes with Theorem 2.4.3: let $\mathbf{g}_0$ be decomposed following Levi-Mal'cev decomposition; then $(\mathbf{p}, J)$ is a LCR-structure if and only if its factors are LCR-structures whose semidirect sum by ad is $(\mathbf{p}, J)$ itself. Obviously this result describes all the LCR-structures. The only indetermination is due to the knowledge of the

ideals of solvable Lie-algebras.
Hence, in Section 2.5 the problem of the existence of Levi-flat CR-structure is solved; and their description is given in the terms of a new Lie-product Γ on $\mathbf{p}$.

2.2 Semisimple LCR-structures.

In this Section we denote by $\mathbf{g}_0$ a real Lie-algebra and by B its Killing form. The existence and the description of semisimple LCR-structures depend on the compactness of the Lie-algebra. Thus, we study, separately, the compact and the noncompact case. Remind that a Lie-algebra $\mathbf{g}_0$ is *compact* if there exists a compact Lie-group whose Lie-algebra is $\mathbf{g}_0$. That is equivalent to giving the decomposition $\mathbf{g}_0 = \zeta(\mathbf{g}_0) \odot [\mathbf{g}_0, \mathbf{g}_0]$, where $\zeta(\mathbf{g}_0)$ is the center of $\mathbf{g}_0$ and $[\mathbf{g}_0, \mathbf{g}_0]$ is semisimple and compact.

It is a classical fact that the existence of a complex structure on a compact Lie-algebra implies the abelianity of the algebra itself. Moreover, a CR-structure $(\mathbf{p}, J)$ such that $\mathbf{p}$ is in the center of $\mathbf{g}_0$, is trivially a LCR-structure, so we can hopefully expect a CR analogous of the complex result. Such an analogous result is based on the

Lemma 2.2.1 *Given a LCR-structure* $(\mathbf{p}, J)$ *on* $\mathbf{g}_0$, $\mathbf{p}$ *admits a biinvariant metric if and only if* $\mathbf{p}$ *is abelian.*

Proof: a metric g is biinvariant, whenever

$$g([X,Y],Z) = g(X,[Y,Z]),$$

for all X, Y, Z in g_0. Let p be abelian, then any metric is, certainly, biinvariant. In order to prove the converse, we can impose that J is an isometry with respect to g (otherwise we substitute g with $g'(X,Y) \doteq g(X,Y) + g(JX, JY)$). With this hypothesis the following chain of equivalences is true, for any X, Y, Z in p

$$\begin{aligned} g([X,Y],Z) &= g(J[X,Y],JZ) = g([X,JY],JZ) = \\ g(X,[JY,JZ]) &= -g(X,[Y,Z]) = -g([X,Y],Z), \end{aligned}$$

therefore $g([X,Y],Z)$ vanishes. ■

Since any compact Lie-algebra admits a biinvariant metric, we have the

Proposition 2.2.2 *Let g_0 be a compact Lie-algebra, (p, J) is a LCR-structure on g_0 if and only if p is abelian. Moreover, the same result is true when the only p is compact.*

The previous proposition permits us to describe the compact case with the

Theorem 2.2.3 *There are no LCR-structures on a compact semisimple Lie-algebra. Furthermore, when g_0 is a compact Lie-algebra, (p, J) is a LCR-structure on g_0 if and only if p is included in the center $\zeta(\mathrm{g}_0)$.*

Proof: the non-existence of abelian ideals in a semisimple Lie-algebra concludes the first part of the assertion. About the second one. suppose that a compact Lie-algebra $\mathfrak{g}_0$ supports a LCR-structure $(\mathfrak{p}, J)$. then $\mathfrak{p}$ takes the form $\mathfrak{p}_1 \oplus \mathfrak{p}_2$ where $\mathfrak{p}_2$ is an ideal of the Levi-subalgebra $\mathcal{D}\mathfrak{g}_0$ and $\mathfrak{p}_1 = \mathfrak{p} \cap \zeta(\mathfrak{g}_0)$ is the radical of $\mathfrak{p}$. In the case that J maps $\mathfrak{p}_2$ in itself, then $(\mathfrak{p}_2, J|_{\mathfrak{p}_2})$ would be a LCR-structure of $[\mathfrak{g}_0, \mathfrak{g}_0]$, that is impossible. Hence, $\mathfrak{p}$ coincides with $\mathfrak{p}_1$ and stays in $\zeta(\mathfrak{g}_0)$.
Let us conclude proving that J maps, really, $\mathfrak{p}_2$ in itself. Consider the complex subalgebras $\mathfrak{q}_j \doteq \{X - iJX : X \in \mathfrak{p}_j\}$. Obviously it is $\mathfrak{q} = \mathfrak{q}_1 \oplus \mathfrak{q}_2$ and $\mathfrak{q}_2$ is another LCR-structure of $\mathfrak{g}$. Hence, it is given the endomorphism $J_2 : \mathfrak{p}_2 \to \mathfrak{p}_2$. Take $X \in \mathfrak{p}_2$, then $X - iJX$ is in $\mathfrak{q}$, and $X - iJ_2X$ is in $\mathfrak{q}_2$. With a direct computation, we show that $i(J_2X - JX) = (X - iJX) - (X - iJ_2X) = (X + iJ_2X) - (X + iJX) \in \mathfrak{q} \cap \overline{\mathfrak{q}} = \{0\}$, which means that J maps $\mathfrak{p}_2$ in itself. ■

Now we move to the study of LCR-structures on semisimple non-compact Lie-algebras. The simple case is trivial. In fact, since there are no nontrivial ideals, a LCR-structure on a simple Lie-algebra is, really, an *ad*-invariant complex one, if it exists. Moreover, it is well known that a semisimple Lie-algebra is direct sum of simple ideals. These facts bring us to the

Proposition 2.2.4 *A LCR-structure on a semisimple Lie-algebra is completely defined by its simple ideals endowed with a complex structure. Moreover, the same result is true whenever* $\mathfrak{g}_0$ *is a generic Lie-algebra and* $\mathfrak{p}$ *is a semisimple ideal.*

Proof: since $\mathfrak{q}$ is semisimple, $\mathfrak{p} = Re\mathfrak{q}$ is semisimple, too. So, $\mathfrak{p} = \mathfrak{p}_1 \odot \ldots \mathfrak{p}_k$, where the $\mathfrak{p}_j$ are simple ideals of $\mathfrak{p}$. Define $\mathfrak{q}_j \doteq \{X - iJX : X \in \mathfrak{p}_j\}$. Then $\mathfrak{q} = \mathfrak{q}_1 \odot \ldots \odot \mathfrak{q}_k$ and $[\mathfrak{q}_j, \mathfrak{q}] \subseteq \mathfrak{q}_j$. So $\mathfrak{q}_j$ is a CR-structure of $\mathfrak{g}$ which corresponds to the pair $(\mathfrak{p}_j, J_j)$. A trivial computation shows that $J_j = J|_{\mathfrak{p}_j}$. Hence, $J\mathfrak{p}_j \subseteq \mathfrak{p}_j$. This fact concludes the proof. ■

Hence, a LCR-structure on a semisimple Lie-algebra is given by the complex structures on some simple factors. Each of these factors is described in the Cartan's classification of the complex simple Lie-algebras

$\mathfrak{g}$	G	U	$\zeta(\mathrm{U}')$	$dim\mathrm{U}$
$a_n (n \geq 1)$	$\mathbf{SL}(n+1, \mathbf{C})$	$\mathbf{SU}(n+1)$	$\mathbf{Z}_{n+1}$	$n(n+2)$
$b_n (n \geq 2)$	$\mathbf{SO}(2n+1, \mathbf{C})$	$\mathbf{SO}(2n+1)$	$\mathbf{Z}_2$	$n(2n+1)$
$c_n (n \geq 3)$	$\mathbf{Sp}(n, \mathbf{C})$	$\mathbf{Sp}(n)$	$\mathbf{Z}_2$	$n(2n+1)$
$d_n (n \geq 4)$	$\mathbf{SO}(2n, \mathbf{C})$	$\mathbf{SO}(2n)$	$\mathbf{Z}_4, n = odd$ $\mathbf{Z}_2 + \mathbf{Z}_2, n = even$	$n(2n-1)$
e_6	$E_6^{\mathbf{C}}$	E_6	$\mathbf{Z}_3$	78
e_7	$E_7^{\mathbf{C}}$	E_7	$\mathbf{Z}_2$	133
e_8	$E_8^{\mathbf{C}}$	E_8	$\mathbf{Z}_1$	248
f_4	$F_4^{\mathbf{C}}$	F_4	$\mathbf{Z}_1$	52
g_2	$G_2^{\mathbf{C}}$	G_2	$\mathbf{Z}_1$	14

In the Table (cf. [HE]), g is a simple Lie-algebra over C; n the dimension of a Cartan-subalgebra; **G** a connected Lie-group such that $Lie(\mathrm{G}) = \mathrm{g}^{\mathrm{R}}$, where g^{R} is the realification of g; **U** an analytical subgroup such that $Lie(\mathrm{U})$ is a compact real form of g (i.e. **U** is a maximal compact subgroup); and **U**$'$ is the universal covering of U.

Let us summarise the results in the following

Proposition 2.2.5 *Let* g_0 *be a semisimple and noncompact Lie-algebra. Then we give the decomposition* $\mathrm{g}_0 = \mathrm{r}_1 \odot \ldots \odot \mathrm{r}_j \odot \mathrm{p}_1 \odot \ldots \odot \mathrm{p}_h$, *where:*

1. *both* r_i *and* p_i *are simple real ideals;*
2. *on the* r_i *there are no complex structures;*
3. *any* p_i *takes one of the forms in the Table.*

With such a decomposition we may choose any sum $\mathrm{p} = \odot_{l=1}^{k} \mathrm{p}_{i_l}$ *with the endomorphism* $J = J_{i_1} \oplus \ldots J_{i_k}$. *The pair* (p, J) *is the generic LCR-structure on* g_0.

2.3 Solvable LCR-structures.

A real Lie-algebra g_0 is *solvable* if one of its *derived* subalgebras vanishes. Since any ideal of g_0 is solvable, a LCR-structure on g_0 is an *ad*-invariant complex structure on a solvable ideal.

Lemma 2.3.1 *Suppose* g_0 *is a solvable Lie-algebra and* (p, J) *is a LCR-structure. Then there exists a subspace* u *such that* $\mathrm{p} = \mathrm{u} \oplus J\mathrm{u}$

and $J = \begin{pmatrix} 0 & J'' \\ J' & 0 \end{pmatrix}$, *where* J' *and* J'' *are the restrictions of* J *to* $\mathbf{u}$ *and to* $J\mathbf{u}$, *respectively.*

Proof: since $\mathbf{p}$ is solvable there exists an its codimension one ideal $\mathbf{p}_1$ [VA]. It is easy to show that $J\mathbf{p}_1 \neq \mathbf{p}_1$. Then, there exists $X_1 \in \mathbf{p}_1$ such that $\mathbf{p} = L(X_1, JX_1) \oplus \mathbf{p}_1 \cap J\mathbf{p}_1$. Moreover $(\mathbf{p}_1 \cap J\mathbf{p}_1, J)$ is a LCR-structure of $\mathbf{p}$. Now the same fact is true for the pair $(\mathbf{p}_1 \cap J\mathbf{p}_1, \mathbf{p}_2)$, where $\mathbf{p}_2$ is a codimension one ideal in $\mathbf{p}_1 \cap J\mathbf{p}_1$. In that way, we find a family $X_1 \dots X_k$, such that $\mathbf{p} = L(X_1 \dots X_k, JX_1 \dots JX_k)$ and the space $\mathbf{u} = L(X_1 \dots X_k)$ is the desired one. ∎

Let us show the converse: any ideal of a solvable Lie-algebra supports a LCR-structure if and only if it is even dimensional; in that case we write $\mathbf{p}$ as the sum $\mathbf{p} = \mathbf{u} \oplus \mathbf{v}$, where $\mathbf{u}$ and $\mathbf{v}$ have the dimension $\frac{1}{2}\dim\mathbf{p}$. Chosen a linear monomorphism $A : \mathbf{v} \to \mathbf{p}$ such that $\mathbf{u} = A\mathbf{v}$, the complex structure $J = J_A \doteq \begin{pmatrix} 0 & A \\ -A^{-1} & 0 \end{pmatrix}$ is generic: so the LCR-structures depend only on the splitting of $\mathbf{p}$ in equal-dimensional subspaces. Let us proof this fact by induction.
The simplest solvable algebras are the abelian ones, i.e. the ones whose first derived vanishes.

Lemma 2.3.2 *Let* $\mathbf{g}_0$ *be an abelian real Lie-algebra. Then there exists an* ad_X*-invariant complex structure* J *on the ideal* $\mathbf{p}$ *if and only if* $\mathbf{p}$ *is even-dimensional. In that case there exist a linear subspace* $\mathbf{u}$ *and a monomorphism* $A : \mathbf{u} \to \mathbf{p}$ *such that*

1. $\mathfrak{p} = A\mathfrak{u} \oplus \mathfrak{u}$
2. $J = J_A \doteq \begin{pmatrix} 0 & A \\ -A^{-1} & 0 \end{pmatrix}$.

Moreover, fixed $\mathfrak{p}$, *all the LCR-structure* $(\mathfrak{p}, J_A)$ *are equivalent, independently on the subspace* $\mathfrak{u}$ *and on the morphism* A. *Hence, the structure is unique.*

Proof: suppose that $\mathfrak{p}$ is endowed with an ad_X-invariant complex structure J, then Lemma 2.3.1 gives us the pair $(\mathfrak{u}, J')$ desired.
Vice versa, let $\mathfrak{p}$ be an even-dimensional ideal. Then, choose $\mathfrak{u}$ and A, such that $\mathfrak{p} = \mathfrak{u} \oplus A\mathfrak{u}$. The endomorphism J_A is trivially an ad_X-invariant complex structure on $\mathfrak{p}$. If one considers the automorphism $\phi_{AB} \doteq \begin{pmatrix} I & 0 \\ 0 & BA^{-1} \end{pmatrix}$, one has an isomorphism between $(\mathfrak{p}, J_A)$ and $(\mathfrak{p}, J_B)$, in fact $J_A\phi_{AB} = \phi_{AB}J_B$. Hence, the complex structure does not depend on A. Finally, we show that does not depend neither on $\mathfrak{u}$: let $(\mathfrak{v}, C)$ be a pair such that $\mathfrak{p} = \mathfrak{v} \oplus C\mathfrak{v}$. Then we have $\mathfrak{v} = D\mathfrak{u}$ and $\mathfrak{p} = D\mathfrak{u} \oplus AD\mathfrak{u}$, where we have taken D Lie-isomorphism. It is easy to show that the pairs $(D\mathfrak{u} \oplus AD\mathfrak{u}, J_A)$ and $(\mathfrak{u} \oplus D^{-1}AD\mathfrak{u}, J_{D^{-1}AD})$ are isomorphic. ■

In Section 2.2, we have shown that, given a compact Lie-algebra $\mathfrak{g}_0$, $(\mathfrak{p}, J)$ is a LCR-structure if and only if $\mathfrak{p}$ is contained in the center $\zeta(\mathfrak{g}_0)$. Lemma 2.3.2 permits us to describe in a deeper way these LCR-structures. In fact, suppose $(\mathfrak{p}, J)$ is a LCR-structure, then $\mathfrak{p}$ has to take the form $\mathfrak{p} = \mathfrak{u} \oplus A\mathfrak{u}$, with $J = J_A$. Thus, a LCR-structure on a compact Lie-algebra is equivalent to the choice of an even-dimensional linear subspace of the center.

Theorem 2.3.3 *A solvable Lie-algebra* g_0 *admits a unique LCR-structure supported on each its even-dimensional ideal. Let* (p, J) *be a LCR-structure, then there exist two vector spaces* u *and* v *and an isomorphism* A *between* u *and* v *such that* $\mathrm{p} = \mathrm{u} \oplus A\mathrm{u}$ *and* $J = J_A$*. Moreover, fixed* p *all the LCR-structures* (p, J_A) *are equivalent.*

Proof: let k be the minimum integer such that $\mathcal{D}^k \mathrm{g}_0 = 0$, then make the proof by induction over k. The base of the induction is given by the abelian case. Now, let g_0 be a solvable but not abelian real Lie-algebra. In any case, $\mathrm{g}_0{}' \doteq \mathrm{g}_0/\mathcal{D}\mathrm{g}_0$ is abelian. Furthermore J maps $\mathcal{D}\mathrm{p}$ on itself, since $Jad_X = ad_X J$. So the induced morphism J' defines a LCR-complex structure. If we apply the previous Lemma, we have that $\mathrm{p}' = \mathrm{w}' \oplus J'|_{\mathrm{w}'}\mathrm{w}'$ and $J' = J_{J'|_{\mathrm{w}'}}$. Choose a subspace w in the class w', then we obtain $\mathrm{g}_0 = \mathrm{w} \oplus J^+\mathrm{w} \oplus \mathcal{D}\mathrm{g}_0$ and $J = \begin{pmatrix} 0 & J^+ & 0 \\ J^+ & 0 & 0 \\ 0 & 0 & J_1 \end{pmatrix}$, where J^+ is the restriction to w and J_1 the one to $\mathcal{D}\mathrm{g}_0$. Finally, we apply the inductive hypothesis on the pair $(\mathcal{D}\mathrm{g}_0, J_1)$. ■

In conclusion, a solvable Lie-algebra g_0 admits one LCR-structure on each even-dimensional ideal (in the hypothesis that it exists) given by an isomorphism J_A. Hence LCR-structures are essentially given by the choice of even-dimensional ideals. Remark that it is possible to have different LCR-structures of the same dimension.

Example 8 *Let* g_0 *be the real three-dimensional linear space spanned by* (E_1, E_2, E_3) *whose Lie-product is given by*

$[E_1, E_2] = [E_2, E_3] = 0$

$[E_1, E_3] = E_3$.

Consider now the solvable ideals $\mathrm{p}_1 = L(E_2, E_3)$ *and* $\mathrm{p}_2 = L(E_1, E_3)$. *Since* p_2 *is not abelian, the LCR-structures defined on them are inequivalent.*

Example 9 *Let* $\mathrm{g}_0(n)$ *be the set of upper triangular* $n \times n$ *real matrices, and* n_0 *be the ideal whose elements have* 0 *on the diagonal. A trivial computation shows that* n_0 *is nilpotent and it coincides with* $\mathcal{D}\mathrm{g}_0(n)$. *Hence,* $\mathrm{g}_0(n)$ *is solvable. Consider the matrix* E_{ij} *which has* 1 *in* (i, j)*-position and* 0 *elsewhere. If* n_0 *is odd-dimensional, then*

$$\mathrm{n}_k = \mathrm{n}_0 \oplus \oplus_{j=1}^{2k-1} \mathbf{R} E_{i_j i_j}$$

is an even-dimensional ideal, as well as it is

$$\mathrm{n}_k = \mathrm{n}_0 \oplus \oplus_{j=1}^{2k} \mathbf{R} E_{i_j i_j},$$

when n_0 *is even-dimensional. In both the cases,* $\mathrm{g}_0(n)$ *admits at least* 2^{n-1} *LCR-structures, not necessary inequivalent.*

2.4 The Levi-Mal'cev decomposition.

Let p be an ideal of g_0. Then its radical p_r is given by $\mathrm{p} \cap \mathrm{r}$, where r is the radical of g_0. Furthermore, if p_s is an its Levi-subalgebra, there exists a Levi-subalgebra $\mathbf{s}$ of $\mathbf{g}_0$ containing $\mathbf{p_s}$. Thus, there are the two Levi-Mal'cev decompositions: $\mathbf{p} = \mathbf{p_r} \oplus_{ad} \mathbf{p_s}$ and $\mathbf{g}_0 = \mathbf{r} \oplus_{ad} \mathbf{s}$. Since $\mathbf{p_r}$ is the radical of $\mathbf{p}$ it contains both $[\mathbf{p_s}, \mathbf{r}]$ and $[\mathbf{p_r}, \mathbf{s}]$.

Suppose, now, that $(\mathbf{p}, J)$ is a LCR-structure and that J is denoted by the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Moreover, choose the elements U in $\mathbf{r}$, V in $\mathbf{p_r}$, X in $\mathbf{s}$ and Y in $\mathbf{p_s}$. Then, the condition $ad_{U+X}J = Jad_{U+X}$ is equivalent to the following 1) $A[U,V] = [U,AV] + [U,CV]$

2) $A[X.V] = [X, AV]$

3) $A[U,Y] = [U, BY] + [U, DY]$

4) $B[X,Y] = [X, BY]$

5) $C[U,V] = 0$

6) $C[U,Y] = 0$

7) $C[X,V] = [X, CV]$

8) $D[X,Y] = [X, DY]$.

A direct computation shows that J is the direct sum of A and D. In fact, there is the

Proposition 2.4.1 *The matrices B and C vanish.*

Proof: in consequence of 7), ImC is an ideal of $\mathbf{s}$. Moreover, we have that $[CV, CV_1] = C[CV, V_1] = 0$, so ImC is abelian. Thus, it is an abelian ideal of a semisimple Lie-algebra and it has to vanish.

The fourth condition says that $\ker B$ is an ideal of $\mathbf{p_s}$. Hence, it is semisimple: moreover, $\mathbf{p_s}/\ker B$ is semisimple, too. Otherwise, every subspace $\mathbf{t}$ of $\mathbf{r}$ verifies $\mathcal{D}^n\mathbf{t} = 0$, for a suitable n. So ImB does. As linear spaces, we have that $\mathbf{p_s}/\ker B$ and ImB are isomorphic, via the isomorphism $jX^+ \doteq BX$, where $X^+ = X + \ker B \in \mathrm{p_s}/\ker B$. Let us compute the product $[jX^+, jY^+]$.

First of all, take X, Y in $\mathrm{p_s}$, and compute

$$\begin{aligned}[BX, BY] &= A[BX, Y] - [BX, DY] = AB[X, Y] - B[X, DY] = \\ &= -BD[X, Y] - B[X, DY] = -2BD[X, Y].\end{aligned}$$

Furthermore, D sends $\ker B$ in $\ker B$, in fact B intertwines A and $-D$. Hence, $[jX^+, jY^+] = -2j(D[X, Y])^+$. So, we can conclude that $\mathcal{D}^n(\mathrm{p_s}/\ker B)$ vanishes, since $\mathrm{p_s}/\ker B$ is semisimple. Thus, $\mathrm{p_s}$ coincides with $\ker B$. ■

Remark 2.4.2 *The vanishing of C does not depend on the fact that the first factor is solvable. So for a semidirect sum $\mathrm{g_0} \oplus_S \mathrm{g}_0'$, with the second factor g_0' semisimple, a splitted LCR-structure takes the form* $(\mathrm{p} \oplus \mathrm{p}', \begin{pmatrix} A & B \\ 0 & D \end{pmatrix})$.

Proposition 2.4.1 permits us to simplify the list of relations characterising a LCR-structure:

1) $(\mathrm{p_r}, A)$ is a LCR-structure on r

2) $(\mathrm{p_s}, D)$ is a LCR-structure on s

3) $[\mathrm{p_s}, \mathrm{r}] \subset \mathrm{p_r}$

4) $[\mathrm{p_r}, \mathrm{s}] \subset \mathrm{p_r}$,

5) $A[X, V] = [X, AV], \forall X \in \mathrm{s}, V \in \mathrm{p_r}$;

6) $A[U, Y] = [U, DY], \forall U \in \mathrm{r}, Y \in \mathrm{p_s}$.

Theorem 2.4.3 *Let* $\mathfrak{g}_0$ *be a real Lie-algebra. Then, there exists an its Levi-subalgebra* $\mathfrak{s}$ *such that* $(\mathfrak{p}_\mathfrak{r}, J_\mathfrak{r})$ *and* $(\mathfrak{p}_\mathfrak{s}, J_\mathfrak{s})$ *are LCR-structures on* $\mathfrak{r}$ *and* $\mathfrak{s}$*, respectively; and* $(\mathfrak{p}, J)$ *is their semidirect sum by the adjoint derivation. Vice versa, if one considers two LCR-structures* $(\mathfrak{p}_\mathfrak{r}, A)$ *and* $(\mathfrak{p}_\mathfrak{s}, D)$ *which verify*

1) $[\mathfrak{p}_\mathfrak{s}, \mathfrak{r}] \subset \mathfrak{p}_\mathfrak{r}$

2) $[\mathfrak{p}_\mathfrak{r}, \mathfrak{s}] \subset \mathfrak{p}_\mathfrak{r}$

3) $A[X, V] = [X, AV]$

4) $A[U, Y] = [U, DY]$

their semidirect sum by ad is a LCR-structure on $\mathfrak{g}_0$*.* ■

2.5 Levi-flat CR-structures.

Morimoto showed that there always exist complex structures J_{MO} on any even dimensional real reductive Lie-algebra, [MO]. Using this result, we prove the existence of Levi-flat CR-structures on every Lie-algebras (except $\mathfrak{su}(2)$). Next, we study their structure. In order to do this, we introduce a new Lie-product Γ on $\mathfrak{p}$ with respect of which the CR-structure $(\mathfrak{p}, J)$ is a Lie's one. Then, we apply Theorem 2.4.3. This allows to give a general structure theorem for Levi-flat CR-structures (Theorem 2.5.10).

Theorem 2.5.1 *The only Lie-algebra which does not support any Levi-flat CR-structure is* $\mathfrak{su}(2)$.

Proof: consider a Levi-Mal'cev decomposition $\mathrm{g}_0 = \mathrm{r} \oplus_{ad} \mathrm{s}$. When s is even-dimensional, Morimoto assures that there exists a complex structure J_{MO} on it. The pair (s, J_{MO}) is a Levi-flat CR-structure.

Furthermore, we have seen that, if $\dim \mathbf{r} \geq 2$, there exists a solvable Levi-flat CR-structure $(\mathbf{p}, J_A)$ on $\mathbf{r}$.

So, we have to study the case $\dim \mathrm{s}$ odd and $\dim \mathbf{r} \leq 1$. When $\dim \mathbf{r} = 1$, g_0 is reductive. In fact, take an element of the center $R_0 + S_0$. Thus, S_0 vanishes and $[R_0, S] = 0$, for any S in s. Hence. if $\zeta(\mathrm{g}_0) \neq \{0\}$, then $[\mathbf{r}, \mathrm{s}] = 0$, $\mathbf{r} = \zeta(\mathrm{g}_0)$ and $\mathrm{s} = \mathcal{D}\mathrm{g}_0$. Vice versa. suppose that the center vanishes. Then, since $\mathbf{r}$ is an abelian ideal. $[\mathbf{r}, \mathrm{s}]$ is not null and it coincides with $\mathbf{r}$. So, $\mathrm{g}_0 = \mathcal{D}\mathrm{g}_0$. In both cases. $\mathrm{g}_0 = \zeta(\mathrm{g}_0) \odot \mathcal{D}\mathrm{g}_0$. So, g_0 is an even-dimensional reductive Lie-algebra. and there is a J_{MO} complex structure on the whole g_0.

The last case is given by the odd-dimensional semisimple Lie-algebras g_0, and it is divided as follows:

1. If $rank\mathrm{g}_0 \geq 2$, any even-dimensional linear subspace $\mathbf{p}$ of a Cartan subalgebra supports a Levi-flat CR-structure $(\mathbf{p}, J_A)$ on g_0.

2. When $rank\mathrm{g}_0 = 1$, taken a Cartan subalgebra $\mathbf{h} = \mathbf{R}H_\alpha$, the only roots are the vanishing one and $\pm\alpha$. So, the algebra is of the form $\mathrm{g}_0 = \mathbf{R}H_\alpha \oplus \mathbf{R}X_\alpha \oplus \mathbf{R}X_{-\alpha}$, hence it is three-dimensional. Finally. the only three-dimensional semisimple real Lie-algebras are $\mathbf{su}(2)$ and $\mathbf{sl}(s, \mathbf{R})$. In the Appendix of Chapter 1, we have seen that $\mathbf{su}(2)$ has no Levi-flat CR-structure; while $\mathbf{sl}(2, \mathbf{R})$ is endowed with the Levi-flat CR-structures $(\mathbf{p}_{a,\alpha}, J_{a,\alpha})$ ■

Let $(\mathfrak{p}, J)$ be a CR-structure on $\mathfrak{g}_0$. Define the bilinear skewsymmetric form $\Gamma : \mathfrak{p} \times \mathfrak{p} \to \mathfrak{p} : (X, Y) \mapsto [X, Y] - [JX, JY]$.

Lemma 2.5.2 *The bilinear form Γ is a Lie-product on $\mathfrak{p}$. Moreover, the structure J is a complex one invariant with respect to the Γ-adjoint derivations of $\mathfrak{p}$.*

Consider a CR-structure $(\mathfrak{p}, J)$ such that $\mathfrak{q}$ is a solvable complex subalgebra. Then $\mathfrak{p}$ satisfies the condition $\mathcal{D}^l\mathfrak{p} = \{0\}$, for a suitable $l \in \mathbb{N}$. By definition, an element of $\mathcal{D}^k_\Gamma\mathfrak{p}$ is sum of elements of $\mathcal{D}^k\mathfrak{p}$, hence, $\mathcal{D}^l_\Gamma\mathfrak{p}$ vanishes; and therefore $(\mathfrak{p}, \Gamma)$ is a real Γ-solvable Lie-algebra. Applying the results of Section 3 to the Γ-LCR-structure $(\mathfrak{p}, J)$, we have the

Proposition 2.5.3 *Let $\mathfrak{g}_0$ be a real Lie-algebra, and $(\mathfrak{p}, J)$ be a CR-structure, such that $\mathfrak{q}$ is solvable. Then there exist a linear subspace $\mathfrak{u}$ of $\mathfrak{p}$ and a linear monomorphism $E : \mathfrak{u} \to \mathfrak{p}$ such that*

1. $\mathfrak{p} = \mathfrak{u} \oplus E\mathfrak{u}$
2. $J = J_E$.

Moreover, any even-dimensional linear subspace $\mathfrak{p}$ may be written as $\mathfrak{p} = \mathfrak{u} \oplus E\mathfrak{u}$ and admits the complex structure J_E.

Let us complexify the Lie-algebra $(\mathfrak{p}, \Gamma)$. Its complexified linear space is $\mathfrak{q}$ itself, on which we may consider the complex product Γ.

Proposition 2.5.4 *The pair $(\mathfrak{q}, \Gamma)$ is a Lie-algebra.*

In fact, $\Gamma(X - iJX, Y - iJY) = \Gamma(X, Y) - \Gamma(JX, JY) - i\{\Gamma(X, JY) + \Gamma(JX, Y)\} = 2\{\Gamma(X, Y) - iJ\Gamma(X, Y)\}$ is an element of $\mathfrak{q}$. ■

We also have that $\Gamma(X - iJX, Y - iJY) = 2\{[X,Y] - [JX,JY] - iJ([X,Y] - [JX,JY])\} = 2[X - iJX, Y - iJY]$, and, as a trivial consequence, $B_\Gamma = 4B$, where B is the Killing form of the Lie-algebra $(\mathfrak{q}, [,])$. This computation suggests the

Proposition 2.5.5 *The complex subalgebra* $\mathfrak{q}$ *is* Γ*-semisimple if and only if it is semisimple.*

In the last part of this Section we consider a Levi-flat CR-structure $(\mathfrak{p}, J)$. Then in view of a classical result, the subalgebra $\mathfrak{p}$ is semisimple if and only if $\mathfrak{q}$ is it. Hence there is the following

Proposition 2.5.6 *Let* $(\mathfrak{p}, J)$ *be a Levi-flat CR-structure on* $\mathfrak{g}_0$. *Then* $\mathfrak{p}$ *is* Γ*-semisimple if and only if* $\mathfrak{p}$ *is semisimple.*

Such correspondence is not true for simple and Γ-simple Levi-flat CR-structures: a semisimple $\mathfrak{p}$ may be a Γ-simple Lie-algebra. In that case, $\mathfrak{p}$ is one of the complex (Γ-)simple algebras of the Cartan's classification [HE]. Otherwise, it is direct sum (with respect of [,] and with respect of Γ) of Γ-simple Γ-ideals $\mathfrak{s}_i$.

Proposition 2.5.7 *Let* $(\mathfrak{p}, J)$ *be a semisimple Levi-flat CR-structure on* $\mathfrak{g}_0$. *If* $\mathfrak{p}$ *is not* Γ*-simple, there are (not necessary simple) ideals* $\mathfrak{s}_i$ *of* $\mathfrak{p}$ *such that*

1. $\mathfrak{p} = \mathfrak{s}_1 \odot \ldots \odot \mathfrak{s}_k$;
2. *each* $\mathfrak{s}_i$ *supports the* Γ_X*-invariant complex structure* $J_{\mathfrak{s}_i}$. ■

Now, take a Levi-flat CR-structure $(\mathfrak{p}, J)$. Since $(\mathfrak{p}, \Gamma)$ is a Lie-algebra, consider its Levi-Mal'cev decomposition $\mathfrak{p} = \mathfrak{r}_\Gamma \oplus_\Gamma \mathfrak{s}_\Gamma$, where $\mathfrak{r}_\Gamma$ is the Γ-radical and $\mathfrak{s}$ is a Γ-Levi-subalgebra.

Proposition 2.5.8 *The Γ-radical $\mathfrak{r}_\Gamma$ and any Γ-Levi-subalgebra $\mathfrak{s}_\Gamma$ are invariant under J.*

A trivial consequence is the

Corollary 2.5.9 *The pairs $(\mathfrak{r}_\Gamma, J|_{\mathfrak{r}_\Gamma})$ and $(\mathfrak{s}_\Gamma, J|_{\mathfrak{s}_\Gamma})$ are Levi-flat CR-structures on $(\mathfrak{p}, \Gamma)$. The structure $(\mathfrak{p}, J)$ is their semidirect sum by Γ.*

The global result can be stated in the following

Theorem 2.5.10 *Let $(\mathfrak{p}, J)$ be a Levi-flat CR-structure. Consider the Γ-Levi-Mal'cev decomposition $\mathfrak{p} = \mathfrak{r}_\Gamma \oplus_\Gamma \mathfrak{s}_\Gamma$. Then the Γ-radical $\mathfrak{r}_\Gamma$ takes the form $\mathfrak{r}_\Gamma = \mathfrak{u} \oplus E\mathfrak{u}$ and the restriction $J|_{\mathfrak{r}_\Gamma}$ is equivalent to J_E. Furthermore, the Γ-Levi-algebra $\mathfrak{s}_\Gamma$ is direct sum of J-invariant ideals $\mathfrak{s}_i$ of $\mathfrak{s}$ which support Γ_X-invariant complex structures $J_i = J|_{\mathfrak{s}_i}$. So the Levi-flat CR-structure is given by the pair $(\mathfrak{p}, J)$ whose elements are*

$$\mathfrak{p} = (\mathfrak{u} \oplus E\mathfrak{u}) \oplus_{ad} \mathfrak{s}_1 \odot \ldots \odot \mathfrak{s}_k$$

$$J = J_E \oplus J_1 \oplus \ldots \oplus J_k.$$

2.6 Appendix.

In this Appendix we describe LCR-structures on low dimensional Lie-algebras $\mathfrak{g}_0$. First of all, remind that there exist just two different bidimensional Lie-algebras: the abelian one and the Lie-algebra $\mathfrak{h}_0$ of the matrices $\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}$, which is solvable. Both of them are endowed with the complex structure given by the "multiplication by i."

So, the case $\dim \mathfrak{g}_0 = 2$ is solved. Now, let $\dim \mathfrak{g}_0$ be greater than 3. Let us start with $\dim \mathfrak{g}_0 = 3$. Such Lie-algebras are completely classified in [MI]. The classification makes use of the map $\varphi : \mathfrak{g} \to \mathbf{R} : X \mapsto tr(ad_X)$. Since $tr([ad_X, ad_Y]) = 0$, φ is a Lie-homomorphism. The kernel $\mathbf{u} \doteq \ker \varphi$ is an ideal called *unimodular kernel*; $\mathfrak{g}_0$ is said *unimodular* if $\mathfrak{g}_0 = \mathbf{u}$. An important result is given by the

Lemma 2.6.1 *Let $\mathfrak{g}_0$ be an unimodular 3-dimensional Lie-algebra endowed with a scalar product. Then there exists an orthonormal base (E_1, E_2, E_3) such that*

1. $[E_2, E_3] = \lambda_1 E_1$, $[E_3, E_1] = \lambda_2 E_2$ *and* $[E_1, E_2] = \lambda_3 E_3$;
2. $B(X, Y) = -2(\lambda_2\lambda_3 X^1 Y^1 + \lambda_1\lambda_3 X^2 Y^2 + \lambda_1\lambda_2 X^3 Y^3)$.

The 3-dimensional unimodular Lie-algebras are classified by the following relations

1. $\lambda_1 = \lambda_2 = \lambda_3 = 0$
2. $\lambda_1 \neq 0$, $\lambda_2 = \lambda_3 = 0$
3. $\lambda_1\lambda_2 \neq 0$, $\lambda_3 = 0$
4. $\lambda_1\lambda_2\lambda_3 \neq 0$.

Case1: $\mathfrak{g}_0$ is abelian and isomorphic to $\mathbf{R}^3$. Each plane supports a LCR-structure: in fact, let $\mathfrak{p} = L(X,Y)$ be a fixed plane; a LCR-structure is given by $J(X,Y) \doteq (-Y,X)$.

Case2: the Lie-product is described by $[E_2,E_3] = \lambda_1 E_1$, $[E_3,E_1] = 0$ and $[E_1,E_2] = 0$. The planes $\mathfrak{p}_2 = L(E_1,E_3)$, $\mathfrak{p}_3 = L(E_1,E_2)$ and $\mathfrak{p}_X = L(E_1,X)$ are abelian ideals endowed with the LCR-structures $J_2(E_1,E_3) \doteq (-E_3,E_1)$ and $J_3(E_1,E_2) \doteq (-E_2,E_1)$. They are all the Levi-flat CR-structures of the algebra.

Case3: let us consider the bidimensional subalgebras: $L(E_1,E_2)$ is the only abelian one and it is even an ideal. Then, we have to look for the solvable ones: so, we study the equation $[X,Y] = Y$.

Since $[X,Y] = \lambda_1(X^2Y^3 - X^3Y^2)E_1 + \lambda_2(X^3Y^1 - X^1Y^3)E_2$, it must be $Y^3 = 0$ and $Y^1Y^2X^3 \neq 0$. Two subcases are possible: or $\lambda_1\lambda_2 > 0$, and there are no solvable subalgebras; either $\lambda_1\lambda_2 < 0$. Hence a solvable subalgebra $p = L(X,Y)$ is generated by $Y = (1, \sqrt{-\frac{\lambda_2}{\lambda_1}}, 0)$ and $X = (X^1, X^2, \sqrt{-\frac{1}{\lambda_1\lambda_2}})$. Since, $[X,E_1] = \sqrt{-\frac{\lambda_2}{\lambda_1}}E_2$, no $L(X,Y)$ is an ideal. In fact, it would be $(0, \sqrt{-\frac{\lambda_2}{\lambda_1}}, 0) = \alpha Y + \beta X$ that implies $\alpha = \beta = 0$, which is a contradiction.

Case4: B is nonsingular, i.e. $\mathfrak{g}_0$ is semisimple. But 3-dimensional semisimple Lie-algebras are simple. Hence $\mathfrak{g}_0$ has no nontrivial ideals. So there are no LCR-structures on such a $\mathfrak{g}_0$. A deeper analysis shows that if all the λ_i are positive, $\mathfrak{g}_0$ is isomorphic to $\mathbf{su}(2)$; while if one of them is negative it is isomorphic to $\mathbf{sl}(2,\mathbf{R})$. In both the cases $\mathfrak{g}_0$ is a real form (compact or not) of $\mathbf{sl}(2,\mathbf{C})$. A detailed study of these Lie-algebras has been done in the Appendix of Chapter 1.

The last case is when $\mathfrak{g}_0$ is not unimodular. Which means that φ is a nonvanishing real linear form. So its kernel $\mathbf{u}$ is an abelian 2-dimensional ideal. And at least one LCR-structure exists.

Summarising all the case, one obtains that a 3-dimensional real Lie-algebra $\mathfrak{g}_0$ either is a (simple) real form of $\mathfrak{sl}(2,\mathbf{C})$ either is endowed with (at least) one LCR-structure given on a 2-dimensional abelian ideal.

Remark that, if one considers the 4-dimensional case, the only non-solvable Lie-algebra endowed with a LCR-structure is $\mathbf{R} \oplus \mathfrak{s}_0$, where $\mathfrak{s}_0$ is a real form of $\mathfrak{sl}(2,\mathbf{C})$. The study of LCR-structures on 2- and 3-dimensional Lie-algebras, make easy the classification on 5-dimensional ones. Such a study is quite interesting since it makes use of Levi-Mal'cev decomposition. In the sequel, let $\dim \mathfrak{g}_0 = 5$. Suppose that $\mathfrak{g}_0$ is decomposed as $\mathfrak{g}_0 = \mathfrak{r}_0 \oplus_{ad} \mathfrak{s}_0$. Let us consider the dimension of $\mathfrak{r}_0$.
When $\dim \mathfrak{r}_0 = 0$, $\mathfrak{g}_0$ is semisimple. Since there are no semisimple algebras of dimension 1 and 2, $\mathfrak{g}_0$ may not have nonvanishing ideals. So $\mathfrak{g}_0$ is simple and it has no LCR-structures.
Let $\dim \mathfrak{r}_0 = 1$. Then $\mathfrak{r}_0$ is the real line and it is abelian; hence $\mathfrak{s}_0$ is simple. So $\mathfrak{g}_0$ has no LCR-structures.
In the case $\dim \mathfrak{r}_0 = 2$, $\mathfrak{r}_0$ either is abelian or it is the solvable algebra of matrices $\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}$. The corresponding Levi-subalgebra $\mathfrak{s}_0$ is simple and coincides either with $\mathbf{su}(2)$ or with $\mathfrak{sl}(2,\mathbf{R})$. Even in this case, $\mathfrak{s}_0$ does not admit LCR-structures. The only one is given by the solvable ideal $\mathfrak{r}_0$ endowed with an endomorphism of the form J_A.
The cases $\dim \mathfrak{r}_0 = 3, 4$ can not occur, since $\mathfrak{s}_0$ should be 2- or 1-

dimensional.

The last case is $\dim \mathfrak{r}_0 = 5$. Then $\mathfrak{g}_0$ is solvable and it admits LCR-structures on all its 2- and 4-dimensional ideals.

Chapter 3

LCR-algebras.

3.1 Introduction to Chapter 3.

In this Chapter (as well in the next one), we focus our attention on LCR-algebras. Precisely, we are interested to describe in what extent the properties of an algebra $\mathbf{g} = \mathbf{g}_0 \oplus_{\mathbf{R}} \mathbf{C}$ depend upon the datum of a LCR-structure $\mathbf{q}$. This is slightly different from what we did in the first two chapters, where a LCR-structure was studied for itself.

Thus, we develop a structure theory of LCR-algebras. First of all, we introduce some useful classes of such Lie-algebras: the CR-nilpotent, the CR-solvable and the CR-semisimple ones.

To study the CR-nilpotent LCR-algebras, we need to define the LCR-representations, i.e. those representations which preserve the LCR-structure. Via these representations, we are able to show that the CR-nilpotent LCR-algebras are characterised by the vanishing of $\mathbf{q} \cap \mathcal{C}^k\mathbf{g}$, for a suitable k. Thus, they are CR-solvable.

Then, in the theory of CR-solvable LCR-algebras the CR-solvable CR-radical $\mathbf{r}^*$ is studied; of course $\mathbf{r}^*$ plays the role of the classical solvable radical. For instance, the property $\mathfrak{r}^* = 0$ determines CR-semisimple LCR-algebras. Moreover, its behaviour is described by the Cartan's criteria for CR-solvability and CR-semisimplicity. In Section 3.7, we give a description of CR-maximal CR-semisimple LCR-algebras $\mathfrak{g}$, where CR-maximal means that any nontrivial LCR-ideal of $\mathfrak{g}$ is contained in $\tilde{\mathfrak{q}}$. A CR-maximal CR-semisimple LCR-algebra is a reductive Lie-algebra and it is a fundamental factor of a CR-semisimple LCR-algebra (Theorem 3.7.4). Thus, we give a structure result for CR-semisimple LCR-algebras. In particular, Theorem 3.7.10 assures that a Lie-algebra $\mathfrak{g}$ admits a semisimple LCR-structure $\overline{\mathfrak{q}}$ if and only if $\mathfrak{g}$ is a noncompact reductive Lie-algebra. Finally, we obtain a result concerning any LCR-algebra and we prove the existence of Levi sub-LCR-algebras $\mathfrak{s}^*$, , obtaining the Levi-Mal'cev CR-decomposition $\mathfrak{g} = \mathfrak{r}^* \oplus_{ad} \mathfrak{s}^*$. Thus, a generic LCR-algebra may be studied as the semidirect sum of a CR-solvable ideal and a CR-semisimple subalgebra.

3.2 CR-nilpotent LCR-algebras.

Let $\mathfrak{g}_0$ be a real Lie-algebra on which a LCR-structure is given via an ideal $\mathfrak{q}$ of the complexified $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbf{R}} \mathbf{C}$. The datum of the real Lie-algebra $\mathfrak{g}_0$ corresponds to a fixed conjugation τ. Consider now a complex linear space V decomposed as $V = W \oplus \overline{W} \oplus V_1$, where the overlined objects are conjugated with respect of its conjugation τ_V.

Definition 3.2.1 *A representation $\rho : \mathrm{g} \to gl(V)$ is said to be a LCR-representation if*

i) $\rho(x)$ commutes with τ_V, for all $x \in \mathrm{g}_0$;

ii) the family $\rho(\mathrm{q})$ maps V into W;

iii) the subspace W is $\rho(\mathrm{g})-$invariant.

A LCR-representation ρ is said to be trivial, whenever $\rho(\mathrm{q})$ vanishes.

A LCR-representation intertwines the conjugation of g and the one of $\mathrm{gl}(V)$: $\rho(\overline{x}) = \overline{\rho(x)}$, $\forall x \in \mathrm{g}$. Moreover the family $\rho(\overline{\mathrm{q}})$ sends V into $\overline{W}$ This implies that ρ sends q in another LCR-structure,

Proposition 3.2.2 *The subalgebra $\rho(\mathrm{q})$ is a LCR-structure on $\rho(\mathrm{g}_0)$. Furthermore, it is a Levi-flat CR-structure on $\mathrm{gl}(V_0)$.*

Proof: since ρ is a representation, $\rho(\mathrm{q})$ is an ideal of $\rho(\mathrm{g})$. Moreover $\overline{\rho(\mathrm{q})} = \rho(\overline{\mathrm{q}})$. In fact, if we take φ in $\rho(\mathrm{q}) \cap \rho(\overline{\mathrm{q}})$, its range is included in $W \cap \overline{W}$. Then φ vanishes. ■

A simple computation shows that *ad* is a LCR-representation.

Definition 3.2.3 *A LCR-representation ρ is said to be CR-nilpotent if and only if, for any $x \in \mathrm{g}$, exists k such that $\rho(x)^k V \cap W = \{0\}$. A LCR-algebra g is said CR-nilpotent, when ad is a CR-nilpotent LCR-representation.*

The second part of the definition has the following converse.

Proposition 3.2.4 *Let ρ be a CR-nilpotent LCR-representation, then $\rho(\mathrm{g})$ is CR-nilpotent.*

Proof: take x in g. Since $\rho(x)$ sends W into W, the map $\rho(x)|_W$ is nilpotent, as well as $\rho(Q)$ is nilpotent, for all Q in q. So, $ad_{\rho(x)|_W}$: $\mathrm{gl}(W) \to \mathrm{gl}(W)$ is a nilpotent map: i.e. $ad^k_{\rho(x)|_W} = 0$.
If x and y are elements of g such that $ad^k_{\rho(x)}\rho(y) \in \rho(\mathrm{q})$, for a suitable k, then $ad^k_{\rho(x)}\rho(y)$ maps V into W. Thus $ad^{k+h}_{\rho(x)}\rho(y)$ vanishes. ■

Lemma 3.2.5 *Let* g *be a CR-nilpotent LCR-algebra. Then there exists a CR-ideal of codimension one.*

Proof: consider the set $S = \{\mathrm{h} \subseteq \mathrm{g} : [\mathrm{h},\mathrm{h}] \subseteq \mathrm{h}, 0 < \dim \mathrm{h} < \dim \mathrm{g}, \tau\mathrm{h} = \mathrm{h}, \mathrm{h} \cap \mathrm{q} \neq \{0\}\}$. S is not empty. In fact, if $x \in \mathrm{p} = Re\mathrm{q}$, $\mathrm{h}(x) = SA(x, Jx)$ verifies the following relations

a) $\mathrm{h}(x) \subseteq \mathrm{q} \oplus \overline{\mathrm{q}} \subseteq \mathrm{g}$;

b) $\mathrm{h}(x) \cap \mathrm{q} \supseteq \mathbf{C}(x - iJx)$;

c) $\mathrm{h}(x) = \tau\mathrm{h}(x)$.

Take an element h in S of maximal dimension. Then h is CR-nilpotent. Consider the linear space $U = \mathrm{g}/\mathrm{h}$, with the subspace $T = \mathrm{q}/\mathrm{h} \cap \mathrm{q}$, then the following decomposition is given $U = T \oplus \overline{T} \oplus U_1$. Let π : $\mathrm{g} \to U$ denote the canonical projection. Finally, remark that when x is an element of h, ad_x induces an endomorphism $\alpha(x)$ of U. The map $\alpha : \mathrm{h} \to \mathrm{gl}(U)$ is a CR-nilpotent LCR-representation of h: take $x, y \in \mathrm{g}$, then $\alpha(x)^k(y + \mathrm{h}) = ad^k_x y + \mathrm{h}$. Such an element is in T if and only if $ad^k_x y$ is in q. Since h is CR-nilpotent, this fact implies that $\alpha(x)^k U \cap T = \{0\}$. The corresponding restricted representation $\tilde{\alpha} : \mathrm{h} \to \mathrm{gl}(T)$ is nilpotent. Take now an element $t \in T/\{0\}$ such that $\alpha(\mathrm{h})t = 0$. The condition is equivalent to the choice of an element

$Q \in \mathrm{q}/\mathrm{q} \cap \mathrm{h}$ such that $ad_Q \mathrm{h} \subseteq \mathrm{h}$. Thus Q is in $\mathrm{n}(\mathrm{h})/\mathrm{h}$, and $\dim \mathrm{n}(\mathrm{h}) > \dim \mathrm{h}$. Since $\mathrm{n}(\mathrm{h})$ is in S, it coincides with g and h is a CR-ideal. For any $y \in \mathrm{n}(\mathrm{h})/\mathrm{h}$, $\mathrm{h}_y \doteq \mathrm{h} \oplus \mathbb{C}(y+\overline{y})$ is an element of S different of h. So, h_y coincides with g and h has codimension one. ■

Theorem 3.2.6 *Given a CR-nilpotent LCR-representation ρ, the set $V' \doteq \{v \in V : \rho(\mathrm{g})v \cap W = \{0\}\}$ is not vanishing.*

Proof: consider the representation $\tilde{\rho} : \mathrm{g} \to \mathrm{gl}(W) : x \mapsto \rho(x)|_W$. Since ρ is CR-nilpotent, $\tilde{\rho}$ is nilpotent. Hence the set $\{v \in V : \rho(\mathrm{g})v = 0\}$ is nonvanishing. Finally, it is contained in V'. ■

Proposition 3.2.7 *Let T be a τ-stable ρ-invariant linear subspace of V. Define $\tilde{V} = V/T$ and $\tilde{\rho} : \mathrm{g} \to \mathrm{gl}(\tilde{V}) : x \mapsto \widetilde{\rho(x)}$, $\widetilde{\rho(x)}[v] = [\rho(x)v]$. Then $\tilde{\rho}$ is a CR-nilpotent LCR-representation.*

Proof: First of all, remark that $\tilde{\tau}\tilde{v} = \tilde{\tau}(v+T) = \overline{v} + T = \widetilde{\tau v}$. Moreover, if $x \in \mathrm{g}_0$,

$$\widetilde{\rho(x)}\tilde{\tau} = \widetilde{\rho(x)\tau} = \widetilde{\tau\rho(x)} = \tilde{\tau}\widetilde{\rho(x)}.$$

Take, now, $Q \in \mathrm{q}$, then $\widetilde{\rho(Q)}\tilde{v} = \widetilde{\rho(Q)v} \in \widetilde{W}$ and $\widetilde{\rho(Q)}\tilde{V} \subseteq \widetilde{W}$. Obviously, $\widetilde{\rho(x)}\widetilde{W} \subseteq \widetilde{W}$, $\forall x \in \mathrm{g}$. Finally, suppose that $\widetilde{\rho(x)}^k v \in \widetilde{W}$. then $\rho(x)^k v \in W$, which is false. ■

Let ρ be a LCR-representation CR-nilpotent of g on V. Consider a subspace $V_1 \subseteq V$ such that

1. $\tau V_1 = V_1$
2. $\rho(\mathrm{g})V_1 \subseteq V_1$

Such a V_1 exists. In fact, $\forall w$ such that $\rho(\mathrm{g})w = 0$, $W_1 = \mathbf{C}(w+\overline{w}) = \overline{W_1}$ and $\rho(\mathrm{g})W_1 = 0$.

Then define the subspaces $V_i = \{v : \rho(\mathrm{g})v \subseteq V_{i-1}\}$

Corollary 3.2.8 *The representation $\rho_i : \mathrm{g} \to \mathrm{gl}(V) : x \mapsto \widetilde{\rho(x)|}_{V_{i+1}}$ is a CR-nilpotent LCR-representation.*

Proposition 3.2.9 *Take the subspaces V_i defined as above. Then there exists an integer s, such that $V_1 \subseteq V_2 \subseteq \ldots \subseteq V_s = V$. For each $i \leq s$, $\tau V_i = V_i$ and V_i is invariant under $\rho(\mathrm{g})$.*

Proof: let us prove by induction that $V_i \subseteq V_{i+1}$. Since $\rho(\mathrm{g})V_1 \subseteq V_1$, then $V_1 \subseteq V_2$. Now, by induction hypothesis, let $V_i \subseteq V_{i+1}$ and take $v \in V_{i+1}$, so $\rho(\mathrm{g})v \subseteq V_i \subseteq V_{i+1}$, and hence $v \in V_{i+2}$. This fact implies that $\rho(\mathrm{g})V_i \subseteq V_i$. Then, we prove that $\tau V_i = V_i$. In fact $\tau V_1 = V_1$; suppose $\tau V_i = V_i$ and take v in V_{i+1}, then $\rho(x)\tau v = \tau\rho(\overline{x})v \in \tau V_i = V_i$. By Corollary 3.2.8, there exists an element $\tilde{v} \in V_{i+1}/V_i$ such that

1. $\tilde{v} \neq 0$
2. $\tilde{\rho}(\mathrm{g})\tilde{v} \cap \widetilde{W} = \{0\}$,

where $\widetilde{W} = W \cap V_{i+1}/W \cap V_i$. Hence, there exists $v \in V$ which does not stay in V_i and such that $\rho(\mathrm{g})v \cap W \cap V_{i+1} \subseteq V_i$. Then $\rho(\mathrm{g})v \cap W \subseteq V_i$ and $v \in V_{i+1}$. So $\dim V_i < \dim V_{i+1}$ and there exists an integer s such that $V_s = V$. ■

If g is CR-nilpotent, then *ad* is a CR-nilpotent LCR-representation. Let us consider a τ-stable ideal $\mathrm{g}_1 \subseteq \mathrm{g}$ which does not intersect $\mathbf{q}$ and take the corresponding family of subspaces $\mathrm{g}_i = \{x : [x, \mathrm{g}] \subseteq \mathrm{g}_{i-1}\}$.

Then, each $\mathfrak{g}_i$ is a τ-stable ideal of $\mathfrak{g}$; there exists an integer s such that $\mathfrak{g}_s = \mathfrak{g}$; and $\mathfrak{g}_i$ is strictly contained in $\mathfrak{g}_{i+1}$. Moreover, for a suitable j, $\mathfrak{g}_j$ is a LCR-ideal.

At this point, we have all the elements to give a characterisation of CR-nilpotent LCR-algebras in the terms of its central series.

Theorem 3.2.10 *The LCR-algebra $\mathfrak{g}$ is CR-nilpotent if and only if there exists p such that $\mathcal{C}^p\mathfrak{g} \cap \mathfrak{q} = \{0\}$.*

Proof: suppose $\mathcal{C}^p\mathfrak{g} \cap \mathfrak{q} = \{0\}$. Since ad_x^p has range in $\mathcal{C}^p\mathfrak{g}$, the intersection $ad_x^p\mathfrak{g} \cap \mathfrak{q}$ vanishes, for all x in $\mathfrak{g}$. Vice versa, consider the above family $\mathfrak{g}_i$. It results that $\mathcal{C}^i\mathfrak{g} \subseteq \mathfrak{g}_{s-i}$, so $\mathcal{C}^{s-1}\mathfrak{g} \cap \mathfrak{q} = \{0\}$. ∎

Corollary 3.2.11 *Let $\mathfrak{g}$ be a n-dimensional CR-nilpotent LCR-algebra, and $\mathfrak{q}$ have codimension k. Then there exist some ideals $\mathfrak{h}_i$ of $\mathfrak{g}$ such that*

1) $\dim \mathfrak{h}_i = n - i$;

2) $\mathfrak{h}_0 = \mathfrak{g} \supseteq \mathfrak{h}_1 \supseteq \ldots \supseteq \mathfrak{h}_m = \{0\}$;

3) $[\mathfrak{g}, \mathfrak{h}_i] \subseteq \mathfrak{h}_{i+1}$;

4) $\mathfrak{h}_i$ *is a LCR-ideal, if* $i \leq k$.

Proof: let $\mathfrak{g}_1 \subseteq \mathfrak{g}_2 \subseteq \ldots \subseteq \mathfrak{g}_s = \mathfrak{g}$ be the elements of the above family. Take a pair of linear subspaces $\mathbf{a}$ and $\mathbf{b}$ such that $\mathfrak{g}_i \subseteq \mathbf{b} \subseteq \mathbf{a} \subseteq \mathfrak{g}_{i+1}$. Then, we have $[\mathfrak{g}, \mathbf{a}] \subseteq [\mathfrak{g}, \mathfrak{g}_{i+1}] \subseteq \mathfrak{g}_i \subseteq \mathbf{b} \subseteq \mathbf{a}$, and we complete the family $\mathfrak{g}_i$ with elements whose codimensions have step 1. ∎

3.3 CR-solvable LCR-algebras.

A sub-LCR-algebra h is said *CR-solvable* if there exists an integer $l > 0$ such that $\mathcal{D}^l\mathrm{h} \cap \mathrm{q} = \{0\}$ and $\mathcal{D}^{l-1}\mathrm{h} \cap \mathrm{q} \neq \{0\}$. Thus the LCR-structure $\mathrm{h} \cap \mathrm{q}$ on h_0 is solvable. Moreover, if h is a solvable sub-LCR-algebra, it is trivially CR-solvable. Thanks to Theorem 3.2.10 a CR-nilpotent LCR-algebra is CR-solvable.

Proposition 3.3.1 *The LCR-algebra* g *is CR-solvable if and only if there exists a family of LCR-ideals* $\mathrm{g}_0 = \mathrm{g}, \mathrm{g}_1, \ldots, \mathrm{g}_s$ *such that*

1. $\mathrm{g}_s \cap \mathrm{q} = \{0\}$
2. $\mathrm{g}_{i+1} \subseteq \mathrm{g}_i$
3. $\mathrm{g}_i/\mathrm{g}_{i+1}$ *is CR-abelian.*

Proof: let g be CR-solvable, then the family $\mathcal{D}^i\mathrm{g}$ is as above. Vice versa, let $\{\mathrm{g}_i\}_{i\in I}$ be a family of LCR-ideals which satisfy the three conditions. Since $\mathrm{g}_i/\mathrm{g}_{i+1}$ is CR-abelian, then $\mathcal{D}^j\mathrm{g} \cap \mathrm{q} \subseteq \mathrm{g}_j \cap \mathrm{q}$; and g is CR-solvable. ■

Theorem 3.3.2 *Let* g *be a CR-solvable LCR-algebra and* r *be its radical. Then* q *is a LCR-structure of* r *and it is given the decomposition* $\mathrm{g} = \tilde{\mathrm{q}} \oplus \mathrm{r}_1 \oplus \mathrm{s}$, *where* $\tilde{\mathrm{q}}$ *is the sum* $\mathrm{q} \odot \overline{\mathrm{q}}$, $\mathrm{r} = \tilde{\mathrm{q}} \oplus \mathrm{r}_1$ *is the decomposition induced by the LCR-structure* q *on* r *and* s *is a Levi-subalgebra.*

Proof: since g is CR-solvable, q is a solvable ideal. Hence, $\mathrm{q} \subseteq \mathrm{r}$. ■ Moreover, we know, by Theorem 2.4.3, that a LCR-structure q on the radical r is a LCR-structure on the whole g if and only if there exists a Levi-subalgebra s, under which it is invariant. Thus, we give the

Theorem 3.3.3 *The LCR-structures with respect of which* $\mathfrak{g}$ *is a CR-solvable LCR-algebra are all the LCR-structures on the solvable radical* $\mathfrak{r}$ *which are invariant under a suitable Levi-subalgebra* $\mathfrak{s}$.

Any subalgebra $\mathfrak{k}$ of a CR-solvable LCR-algebra $\mathfrak{g}$ satisfies the condition $\mathcal{D}^l\mathfrak{k} \cap \mathfrak{q} = \{0\}$. Of course, if it is a sub-LCR-algebra it is CR-solvable. A CR-quotient is CR-solvable, too.

Proposition 3.3.4 *Let* $\mathfrak{h}$ *be a CR-solvable LCR-ideal and* $\mathfrak{g}/\mathfrak{h}$ *be CR-solvable, then* $\mathfrak{g}$ *is CR-solvable.*

Proof: since $\mathfrak{g}/\mathfrak{h}$ is CR-solvable, $\mathfrak{q}/\mathfrak{h}\cap\mathfrak{q}$ is solvable; similarly, $\mathfrak{h}\cap\mathfrak{q}$ is solvable. Thus, $\mathfrak{q}$ is solvable. Let us give the proof by induction on $\dim\mathfrak{g}$. When $\mathfrak{g}$ is bidimensional, it is solvable and it is CR-solvable with respect of its unique LCR-structure. Now, suppose that the fact is true for all the LCR-algebras whose dimension is less than $\dim\mathfrak{g}$. Since $\mathfrak{g}/\mathfrak{h}$ is CR-solvable, $\mathfrak{g}/\mathfrak{h}$ is different from $\mathcal{D}(\mathfrak{g}/\mathfrak{h})$. Thus $\mathfrak{g} \neq \mathcal{D}\mathfrak{g}$. Take a τ-stable subspace of $\mathfrak{g}$ $\mathfrak{k}$ containing $\mathcal{D}\mathfrak{g}$ such that codim$\mathfrak{k}/\mathfrak{h}$ is 1. Then $\mathfrak{k}+\mathfrak{h}$ is a LCR-ideal of codimension 1 of $\mathfrak{g}$. Moreover, $\mathfrak{h}$ is a CR-solvable LCR-ideal of $\mathfrak{k}+\mathfrak{h}$ such that $\mathfrak{k}+\mathfrak{h}/\mathfrak{h}$ is a CR-solvable LCR-ideal of $\mathfrak{g}/\mathfrak{h}$. Thus, $\mathfrak{k}+\mathfrak{h}$ is CR-solvable. Furthermore $\mathfrak{k}+\mathfrak{h}$ contains $\mathcal{D}\mathfrak{g}$. Then, either $\mathcal{D}\mathfrak{g}\cap\mathfrak{q}$ vanishes or $\mathcal{D}\mathfrak{g}$ is a LCR-ideal. In any case $\mathfrak{g}$ is CR-solvable. ∎

Proposition 3.3.5 *Let* $\mathfrak{g}$ *be a CR-solvable LCR-algebra, then there exists a LCR-ideal* $\mathfrak{h}$ *such that* $\dim(\mathfrak{g}/\mathfrak{h}) = 1$

Proof: if $\mathfrak{g}$ is CR-abelian, any τ-stable subspace containing $\tilde{\mathfrak{q}}$ is a LCR-ideal. Otherwise, any τ-stable subspace containing $\mathcal{D}\mathfrak{g}$ is it. Such a subspace exists, since, if $\mathfrak{a} \supseteq \mathcal{D}\mathfrak{g}$, then $\mathfrak{a} + \overline{\mathfrak{a}} \supseteq \mathcal{D}\mathfrak{g}$. ■

Proposition 3.3.6 *Let* $\mathfrak{g}$ *be a CR-solvable LCR-algebra and* ρ *an its LCR-representation on the linear space* V $(\dim_{\mathbb{C}} V = N)$*. Then, there exist some* $\lambda_i \in \mathfrak{g}^*$ *and a basis* $\{v_1 \ldots v_N\}$ *for* V *such that, for any* $x \in \mathfrak{g}$,

$$\rho(x) = \begin{pmatrix} \lambda_1(x) & \ldots & * \\ \vdots & \ddots & \vdots \\ 0 & \ldots \ \lambda_N(x) & \end{pmatrix}.$$

In particular, $\forall x \in \mathfrak{g}, \rho(x)v_1 = \lambda_1(x)v_1$.

The proof, by induction on $\dim \mathfrak{g}$, is based on the following Lemmas 3.3.7 and 3.3.8. The basis of the induction is given by the case $\dim \mathfrak{g} = 2$, for which $\mathfrak{g}$ is solvable and the result is classical, [VA].

Lemma 3.3.7 *In the above hypothesis, there exists a nonvanishing vector of* V *which is an eigenvector for any* $\rho(x)$, $x \in \mathfrak{g}$.

Proof: let $\mathfrak{h}$ be a LCR-ideal of $\mathfrak{g}$ with $\dim(\mathfrak{g}/\mathfrak{h}) = 1$, and x_0 be in $\mathfrak{g}$ such that x_0 is not in $\mathfrak{h}$. By induction hypothesis, consider a nonvanishing vector $w \in V$ and a $\lambda \in \mathfrak{h}^*$ such that $\rho(y)w = \lambda(y)w$, for any $y \in \mathfrak{h}$. Define $w_s \doteq \rho(x_0)^s w$. Let p be the greatest integer such that $w, w_1, \ldots, w_s$ are linearly independents. Define $W_{-1} = \{0\}$ and $W_r = L(w, \ldots w_r)$. Hence, $w_q \in W_p$, whenever $q \geq p$. Moreover, $\rho(x_0)$ maps W_p in itself and W_r into W_{r+1}, where $r < p$. ■

Lemma 3.3.8 *Let $r \leq p$ and $y \in \mathbf{h}$, then $\rho(y)w_r \equiv \lambda(y)w_r, mod W_{r-1}$. Moreover, $\rho(y)W_p \subseteq W_p, \forall y \in \mathbf{h}$.*

Proof: when $r = 0$, we have $\rho(y)w = \lambda(y)w$. Let the thesis be true for $r < p$, then $\rho(y)w_{r+1} = \rho(y)\rho(x_0)w_r = \rho(x_0)\rho(y)w_r + \rho([y, x_0])w_r$. Then $\rho([y, x_0]w_r$ is in W_r and $\rho(y)w_r = \lambda(y)w_r + w'_r$. Thus, $\rho(y)w_{r+1}$ coincides with $\lambda(y)w_{r+1}$ modulo an element of W_r. ■

Let us return to the proof of Theorem 3.3.6: we have shown that $\rho(y)$ and $\rho(x_0)$ let W_p invariant, so $tr(\rho([y, x_0])|_{W_p})$ is null. Otherwise, $\forall z \in \mathbf{h}$, $tr(\rho(z)|_{W_p}) = (1 + p)\lambda(z)$. Hence, $\lambda([y, x_0]) = 0$; so, by induction, $\rho(y)w_r = \lambda(y)w_r$, with y in $\mathbf{h}$. Take, now, an eigenvector v_1 of $\rho(x_0)$ in W_p: $\rho(x_0)v_1 = cv_1$. Define λ_1 as λ on $\mathbf{h}$ and as c in x_0. Obviously, λ_1 stays in $\mathbf{g}^*$ and $\rho(x)v_1 = \lambda_1(x)v_1$, $\forall x \in \mathbf{g}$. Considering the LCR-representation ρ_1 induced by ρ, $\rho_1 : \mathbf{g} \to \mathbf{gl}(V/\mathbf{C}v_1)$, and using the induction on $\dim V$, we obtain the desired basis $\{v_j\}$. ■

Proposition 3.3.9 *Let $\mathbf{g}$ be a CR-solvable LCR-algebra, then there exists a family of sub-LCR-algebras $\mathbf{g}_1 = \mathbf{g}, \mathbf{g}_2, \ldots, \mathbf{g}_{n+1} = \{0\}$, ($n = \dim \mathbf{g}$), such that $\mathbf{g}_{i+1}$ is a 1-codimensional LCR-ideal of $\mathbf{g}_i$.*

Proof: let us construct the LCR-ideal $\mathbf{g}_2$. In the case that $\mathcal{D}\mathbf{g}$ is a LCR-ideal, a τ-stable hyperplane V containing $\mathcal{D}^1\mathbf{g}$ may be chosen as $\mathbf{g}_2$. When $\mathcal{D}\mathbf{g}$ is not a LCR-ideal, $\mathbf{g}$ is CR-abelian. Since $\dim \mathcal{D}\mathbf{g} < n - 1$ (otherwise, it would be $\dim \tilde{\mathbf{q}} = 1$), then as $\mathbf{g}_2$ take a τ-stable hyperplane which contains $\mathcal{D}^1\mathbf{g}$ and which intersects $\mathbf{q}$. Finally, by induction, we construct the family required. ■

Proposition 3.3.10 *Let* g *be a CR-solvable LCR-algebra and* ρ *an its LCR-representation on a finite-dimensional space* V. *Then the set* $\mathrm{a} = \{x \in \mathrm{g} : \rho(x) \textit{is CR-nilpotent}\}$ *is a LCR-ideal containing* $\mathcal{D}\mathrm{g}$.

Proof: consider the sets $\mathrm{b} = \{x \in \mathrm{g} : \rho(x) \text{ is nilpotent}\}$ and $\mathrm{c}(\mathrm{q}) = \{x \in \mathrm{q} : \rho(x) \text{is nilpotent}\}$.
Then $\mathrm{a} \supseteq \mathrm{b} \supseteq \mathrm{c}(\mathrm{q})$; and $\mathrm{a} \cap \mathrm{q} = \mathrm{c}(\mathrm{q}) \supseteq \mathcal{D}\mathrm{q} \neq \{0\}$. Obviously, $\overline{\mathrm{c}(\mathrm{q})} = \mathrm{c}(\overline{\mathrm{q}})$ and $\overline{\mathrm{a} \cap \mathrm{q}} = \mathrm{a} \cap \overline{\mathrm{q}}$.
Since $\rho(x)v_j = \lambda_j(x)v_j mod \oplus_{i<j} \mathbf{C}v_i$, the element x stays in b if and only if $\lambda_i(x) = 0$, for any i. Hence, $\mathcal{D}\mathrm{g} \subseteq \mathrm{b} \subseteq \mathrm{a}$, and a is an ideal containing $\mathcal{D}\mathrm{g}$. So it is a LCR-ideal. ■

Theorem 3.3.11 *The CR-algebra* g *is CR-solvable if and only if* $\mathcal{D}\mathrm{g}$ *is CR-nilpotent.*

Proof: suppose $\mathcal{D}\mathrm{g}$ is CR-nilpotent, then $\mathcal{D}\mathrm{g}$ and $\mathrm{g}/\mathcal{D}\mathrm{g}$ are CR-solvable. Hence g itself is CR-solvable. Vice versa, let g be CR-solvable, then $\mathcal{D}\mathrm{g}$ is contained in the LCR-ideal a, defined in the above Theorem. Thus, $\mathcal{D}\mathrm{g}$ is CR-nilpotent. ■

3.4 The CR-radical.

Take two CR-solvable LCR-ideals h and k. Then, the sum $\mathrm{h} + \mathrm{k}$ is a LCR-ideal and $\mathrm{h} + \mathrm{k}/\mathrm{h} \simeq \mathrm{k}/\mathrm{h} \cap \mathrm{k}$ is CR-solvable. Hence $\mathrm{h} + \mathrm{k}$ is CR-solvable. So, there exists a unique CR-solvable LCR-ideal $\mathrm{r}^* = \mathrm{r}^*(\mathrm{g})$ which contains all the CR-solvable LCR-ideals; r^* is said the *CR-radical* of g.

Proposition 3.4.1 *The LCR-algebra* $\mathfrak{g}$ *is CR-solvable if and only if* $\mathfrak{g}$ *coincides with* $\mathfrak{r}^*$.

Definition 3.4.2 *A LCR-algebra* $\mathfrak{g}$ *is said CR-semisimple if* $\mathfrak{r}^*$ *vanishes.*

Since $\mathfrak{q}$ is an ideal, we know that its radical $\mathfrak{r}(\mathfrak{q})$ is given by the intersection of $\mathfrak{r}(\mathfrak{g})$ with $\mathfrak{q}$, itself. Furthermore, when $\mathfrak{q}$ is a LCR-structure, we have the

Lemma 3.4.3 *The radical* $\mathfrak{r}(\mathfrak{q})$ *is given by the intersection of* $\mathfrak{q}$ *with the CR-radical* $\mathfrak{r}^*(\mathfrak{g})$.

Proof: the intersection $\mathfrak{r}^*(\mathfrak{g}) \cap \mathfrak{q}$ is a solvable ideal of $\mathfrak{q}$, so $\mathfrak{r}^*(\mathfrak{g}) \cap \mathfrak{q} \subseteq \mathfrak{r}(\mathfrak{q}) = \mathfrak{r}(\mathfrak{g}) \cap \mathfrak{q}$. When $\mathfrak{r}(\mathfrak{q})$ vanishes, $\mathfrak{r}^*(\mathfrak{g}) \cap \mathfrak{q}$ vanishes, too. While, when $\mathfrak{r}(\mathfrak{q})$ is not zero, $\mathfrak{r}(\mathfrak{g})$ is a CR-solvable LCR-ideal. Hence, $\mathfrak{r}(\mathfrak{g}) \subseteq \mathfrak{r}^*(\mathfrak{g})$ and the intersections with $\mathfrak{q}$ coincide. ■

The same result is true even for $\tilde{\mathfrak{q}}$.

Lemma 3.4.4 *The intersection* $\mathfrak{r}^* \cap \tilde{\mathfrak{q}}$ *coincides with* $\mathfrak{r}(\tilde{\mathfrak{q}})$. *Moreover,* $\mathfrak{r}(\tilde{\mathfrak{q}})$ *is a LCR-ideal.*

Proof: since $\mathfrak{r}^* \cap \mathfrak{q}$ is solvable, $\mathfrak{r}^* \cap \tilde{\mathfrak{q}}$ is solvable, so $\mathfrak{r}^* \cap \tilde{\mathfrak{q}} \subseteq \mathfrak{r}(\tilde{\mathfrak{q}}) = \mathfrak{r}(\mathfrak{g}) \cap \tilde{\mathfrak{q}}$. Furthermore, $\mathfrak{r}(\tilde{\mathfrak{q}}) \cap \mathfrak{q}$ does not vanish and $\mathfrak{r}(\tilde{\mathfrak{q}})$ is a solvable LCR-ideal. Finally, $\mathfrak{r}(\tilde{\mathfrak{q}}) \subseteq \mathfrak{r}^*$. By the above computation, $\mathfrak{r}(\tilde{\mathfrak{q}})$ is a τ-stable ideal of $\mathfrak{g}$. Otherwise, $\mathfrak{r}(\tilde{\mathfrak{q}}) \cap \mathfrak{q} = \mathfrak{r}^* \cap \mathfrak{q}$ which does not vanish, by definition. So, $\mathfrak{r}(\tilde{\mathfrak{q}})$ is a LCR-ideal. ■

Lemma 3.4.5 *When the CR-radical* $\mathfrak{r}^*$ *is included in the radical* $\mathfrak{r}$, *they coincide.*

Theorem 3.4.6 *The LCR-algebra* $\mathfrak{g}$ *is CR-semisimple if and only if* $\mathfrak{q}$ *is semisimple.*

Proof: the radical of $\mathfrak{q}$ vanishes if and only if the CR-radical of $\mathfrak{g}$ does. ■

When the ideal $\mathfrak{q}$ is semisimple, the direct sum $\tilde{\mathfrak{q}} = \mathfrak{q} \odot \overline{\mathfrak{q}}$ is semisimple, too. The vice versa is also true. Hence, the LCR-algebra $\mathfrak{g}$ is CR-semisimple if and only if $\tilde{\mathfrak{q}}$ is semisimple.

Now, we have all the elements to give a result analogous of Theorem 3.3.2. The LCR-structure of a CR-semisimple LCR-algebra may be seen as the LCR-structure of a semisimple subalgebra, as well as, in that case, the LCR-structure of a CR-solvable LCR-algebra was seen as a LCR-structure of the solvable radical.

Proposition 3.4.7 *Let* $\mathfrak{g}$ *be a CR-semisimple LCR-algebra. Then, there exists a Levi-subalgebra* $\mathfrak{s}$ *which admits* $\mathfrak{q}$ *as LCR-structure and it is given the decomposition* $\mathfrak{g} = \mathfrak{r} \oplus \tilde{\mathfrak{q}} \oplus \tilde{\mathfrak{q}}^{\perp_{\mathfrak{s}}}$, *where* $\tilde{\mathfrak{q}}^{\perp_{\mathfrak{s}}}$ *is the orthogonal of* $\tilde{\mathfrak{q}}$ *with respect to the Killing form of* $\mathfrak{s}$.

Vice versa, by Theorem 2.4.3, a LCR-structure $\mathfrak{q}$ of a Levi subalgebra $\mathfrak{s}$ is a LCR-structure on the whole $\mathfrak{g}$ if $[\mathfrak{q}, \mathfrak{r}]$ vanishes.

Theorem 3.4.8 *The semisimple LCR-structures* $\mathfrak{q}$ *are the LCR-structures of a Levi subalgebra* $\mathfrak{s}$ *which are Levi-flat CR-structures of the centralizer of* $\mathfrak{r}$, $c(\mathfrak{r})$.

Proposition 3.4.9 *The CR-radical* r^* *is invariant under all the CR--derivations; the CR-quotient* g/r^* *is CR-semisimple.*

Proof: a CR-derivation D is an element of $Der(\mathrm{g};\mathrm{q})$, hence $exp(tD)$ is a CR-automorphism and $exp(tD)\mathrm{r}^* = \mathrm{r}^*$, so $D\mathrm{r}^* \subseteq \mathrm{r}^*$.
The projection $\pi : \mathrm{g} \to \mathrm{g}/\mathrm{r}^*$ is a CR-epimorphism. Take a CR-solvable LCR-ideal $\mathrm{h} \subseteq \mathrm{g}/\mathrm{r}^*$. Then $\pi^{-1}(\mathrm{h})$ is a CR-solvable LCR-ideal. So $\mathrm{r}^* \subseteq \pi^{-1}(\mathrm{h}) \subseteq \mathrm{r}^*$, and $\mathrm{h} = \{0\}$, which means that the CR-radical $\mathrm{r}^*(\mathrm{g}/\mathrm{r}^*(\mathrm{g}))$ vanishes. ■

Proposition 3.4.10 *Let* h *be a LCR-ideal. Then* $\mathrm{r}^*(\mathrm{h}) = \mathrm{r}^*(\mathrm{g}) \cap \mathrm{h}$.

Proof: let us consider $[\mathrm{r}^*(\mathrm{h}), \mathrm{g}]$. We may easily compute that it is a CR-solvable LCR-ideal of h. So $[\mathrm{r}^*(\mathrm{h}), \mathrm{g}]$ is included in $\mathrm{r}^*(\mathrm{h})$ and $\mathrm{r}^*(\mathrm{h})$ is a CR-solvable LCR-ideal of g. Hence $\mathrm{r}^*(\mathrm{h})$ is contained in $\mathrm{r}^*(\mathrm{g})$ and $\mathrm{r}^*(\mathrm{h}) \subseteq \mathrm{r}^*(\mathrm{g}) \cap \mathrm{h} \subseteq \mathrm{r}^*(\mathrm{h})$. ■

Theorem 3.4.11 *Let* g *be a CR-semisimple LCR-algebra, then any LCR-ideal is CR-semisimple. Vice versa, if there exists a LCR-ideal* h *containing* q *which, as LCR-algebra, is CR-semisimple, then* g *is CR-semisimple.*

Proof: when $\mathrm{r}^*(\mathrm{g})$ vanishes, by the above Proposition, $\mathrm{r}^*(\mathrm{h})$ vanishes, too. Consider, now, h such that $\mathrm{q} \subseteq \mathrm{h} \subseteq \mathrm{g}$ and h be CR-semisimple. Then q is semisimple and g is CR-semisimple. ■

Let S^* be the set of the LCR-ideals n such that $\rho(x)$ is CR-nilpotent, $\forall x \in \mathrm{n}$. In particular, when n is in S^*, n is an ideal such that $\rho(x)|_W$ is nilpotent.

Take the representation $\rho_W : \mathrm{g} \to \mathbf{gl}(W) : x \mapsto \rho(x)|_W$ with the associated set S_W of the ideals n such that $\rho_W(x)$ is nilpotent. Then $S^* \subseteq S_W$ and, by the existence of the nilradical, there exists an elemente $\mathrm{n}_W \in S_W$ which contains all the elements of S_W. In particular n_W contains all the elements of S^*. Thus $\mathrm{n}_W \cap \mathrm{q}$ does not vanish and $\mathrm{n}_W = \overline{\mathrm{n}_W}$. So, n_W is a LCR-ideal and it is in S^*. Such a result is exposed in the

Proposition 3.4.12 *Given a LCR-algebra* g *and an its finite-dimensional LCR-representation* ρ*, there exists a unique element* $\mathrm{n}^* \in S^*$ *which contains all the elements of* S^**.*

Definition 3.4.13 *A CR-nilideal* m *of* g *is a LCR-ideal such that* ad_x *is CR-nilpotent,* $\forall x \in \mathrm{m}$*. There exists a unique CR-nilideal* n^* *which contains all the CR-nilideal. It is said the CR-nilradical of* g*.*

It is not difficult to show that n^* is contained in r^*; finally any CR-isomorphism of r^* let n^* invariant.

Proposition 3.4.14 *Let* h *be a LCR-ideal, then* $\mathrm{n}^*(\mathrm{h})$ *is a LCR-ideal and coincides with* $\mathrm{n}^*(\mathrm{g}) \cap \mathrm{h}$*.*

The CR-nilradical of g and the one of $\mathrm{r}^*(\mathrm{g})$ coincide. Moreover, we have the

Proposition 3.4.15 *The following equivalences are true:*

1. $\mathrm{n}^*(\mathrm{g}) = \mathrm{n}^*(\mathrm{r}^*(\mathrm{g}))$;
2. $\mathrm{n}^*(\mathrm{g}) = \{x \in \mathrm{r}^* : ad_x \text{ is CR-nilpotent}\}$.

Proof: since $\mathrm{n}^*(\mathrm{g}) \subseteq \mathrm{r}^*(\mathrm{g})$, then $\mathrm{n}^*(\mathrm{g}) \subseteq \mathrm{n}^*(\mathrm{r}^*(\mathrm{g}))$; while $\mathrm{n}^*(\mathrm{r}^*(\mathrm{g}))$ is included in $\mathrm{n}^*(\mathrm{g})$ by definition. The second part of the proof is a consequence of Theorem 3.3.10. ■

Corollary 3.4.16 *If* g *is a CR-solvable LCR-algebra, the CR-nilradical* $\mathrm{n}^*(\mathrm{g})$ *is the set of all the elements* x *such that* ad_x *is CR-nilpotent. Moreover,* $\mathcal{D}\mathrm{g}$ *is contained in* $\mathrm{n}^*(\mathrm{g})$.

Proposition 3.4.17 *Any CR-derivation of* g *maps* r^* *into* n^*. *Hence* $[\mathrm{r}^*, \mathrm{g}] \subseteq \mathrm{n}^*$.

Proof: let $A \in Der^*(\mathrm{g})$ and $\mathrm{g}' = \mathrm{g} \oplus \mathbf{C}$. Define $[(x, c), (x', c')]_A = ([x, x'] + c'Ax - cAx', 0)$. Then $(g', [,]_A)$ is a Lie-algebra; the ideal $\mathrm{q} \oplus \{0\}$ is a LCR-structure of g'; and $\mathrm{r}' = \mathrm{r}^* \oplus \mathbf{C}$ is a CR-solvable LCR-ideal. Moreover, n' is a LCR-ideal of r'. Hence $\mathcal{D}\mathrm{r}' \subseteq \mathrm{n}'$ and $\mathcal{D}\mathrm{r}' \cap (\mathrm{r}^* \oplus \{0\}) \subseteq \mathrm{n}' \cap (\mathrm{r}^* \oplus \{0\}) = \mathrm{n}^* \oplus \{0\}$. Of course, $\mathrm{r}^* \oplus \{0\}$ is an ideal of g' and so. $\forall x \in \mathrm{r}^*, (Ax, 0) = [(x, 0), (0, 1)] \in \mathcal{D}\mathrm{r}' \cap (\mathrm{r}^* \oplus \{0\}) \subseteq \mathrm{n}^* \oplus \{0\}$; which means that $A\mathrm{r}^* \subseteq \mathrm{n}^*$. ■

3.5 Cartan's criteria.

Given a LCR-structure q, an associated representation on $\tilde{\mathrm{q}}$ is introduced, In fact, since $\tilde{\mathrm{q}}$ is an ideal, ad_x maps $\tilde{\mathrm{q}}$ in $\tilde{\mathrm{q}}$, forall x in g. Thus, we define the representation $\psi : \mathrm{g} \rightarrow \mathrm{gl}(\tilde{\mathrm{q}})$ as $\psi(x) \doteq ad_x|_{\tilde{\mathrm{q}}}$.
Hence, there exists a unique maximal ideal n_ψ such that $\psi(x)$ is nilpotent, $\forall x \in \mathrm{n}_\psi$, [VA]. Thanks to Theorem 3.4.12, n^* coincides with n_ψ.

Now, let us consider the symmetric bilinear form

$$B^{\psi}(x,y) = tr(\psi(x), \psi(y)),$$

with the associated ideal

$$\mathrm{g}^{\perp_{\psi}} = \{x \in \mathrm{g} : B^{\psi}(x,y) = 0, \forall y \in \mathrm{g}\}.$$

By a classical result, $[\mathrm{g}^{\perp_{\psi}}, \mathrm{g}] \subseteq \mathrm{n}_{\psi}$. Then, we have the

Lemma 3.5.1 *The CR-nilradical* n_{ψ} *is included in* $\mathrm{g}^{\perp_{\psi}}$.

Proof: take x in n_{ψ}. Then $\psi(x)$ is nilpotent, so $tr(\psi(x)D) = 0$, where D is a derivation of $\tilde{\mathrm{q}}$. In particular, $tr(\psi(x)\psi(y)) = 0$, for all $y \in \mathrm{g}$. ■

Lemma 3.5.2 *When* Q *is an element of* $\tilde{\mathrm{q}}$, *the numbers* $B^{\psi}(x,Q)$ *and* $B(x,Q)$ *coincide, for all* x *in* g.

Proof: first of all, remark that the map $ad_x \circ ad_Q$ sends g into $\tilde{\mathrm{q}}$. Thus, we compute

$$\begin{aligned} B(x,Q) &= tr(ad_x \circ ad_Q) = tr(ad_x \circ ad_Q)|_{\tilde{\mathrm{q}}} = \\ &= tr(ad_x \circ ad_Q|_{\tilde{\mathrm{q}}}) = tr(ad_Q|_{\tilde{\mathrm{q}}} \circ ad_x) = \\ &= tr(ad_Q|_{\tilde{\mathrm{q}}} \circ ad_x)|_{\tilde{\mathrm{q}}} = tr(ad_Q|_{\tilde{\mathrm{q}}} \circ ad_x|_{\tilde{\mathrm{q}}}) = \\ &= B^{\psi}(x,Q). \quad ■ \end{aligned}$$

Now, we have all the elements to proof the Cartan's criteria.

Theorem 3.5.3 *The LCR-algebra* $\mathfrak{g}$ *is CR-solvable if and only if the expression* $B^{\psi}(x,[y,z])$ *vanishes identically.*

Proof: suppose that $\mathfrak{g}$ is CR-solvable. Then $\mathcal{D}\mathfrak{g}$ is a subset of the CR-nilradical $\mathfrak{n}^*$, which is contained in $\mathfrak{g}^{\perp_\psi}$. So $B^{\psi}(x,[y,z]) = 0, \forall x,y,z \in \mathfrak{g}$.

Vice versa consider the case in which $B^{\psi}(x,[y,z])$ vanishes identically. Then, $\mathcal{D}\mathfrak{g}$ is contained in $\mathfrak{g}^{\perp_\psi}$ and $\mathcal{CD}\mathfrak{g} = [\mathcal{D}\mathfrak{g}, \mathcal{D}\mathfrak{g}] \subseteq [\mathfrak{g}^{\perp}, \mathfrak{g}] \subseteq \mathfrak{n}_{\psi} = \mathfrak{n}^*$. So $\mathcal{CD}\mathfrak{g}$ is a CR-nil-ideal. Thus, $\mathcal{D}\mathfrak{g}$ is a CR-nil-ideal, and $\mathfrak{g}$ is CR-solvable. ■

Theorem 3.5.4 *The LCR-algebra* $\mathfrak{g}$ *is CR-semisimple if and only if* B^{ψ} *is nonsingular.*

Proof: in an equivalent way, we shall show that $\mathbf{r}^* \neq \{0\}$ if and only if $\mathfrak{g}^{\perp_\psi} \neq \{0\}$.

Let $\mathbf{r}^*$ do not vanish. When $[\mathbf{r}^*, \mathfrak{g}] \neq \{0\}$, then $\mathfrak{g}^{\perp_\psi}$ does not vanish. In fact, it contains $\mathfrak{n}^*$ which contains $[\mathbf{r}^*, \mathfrak{g}]$; otherwise $[\mathbf{r}^*, \mathfrak{g}] = \{0\}$ means that $\mathbf{r}^*$ is contained in the centre of $\mathfrak{g}$, $\zeta(\mathfrak{g})$. In particular, $\mathbf{r}^*$ coincides with $\zeta(\mathfrak{g})$ and then $\mathfrak{g}^{\perp_\psi} \supseteq \mathfrak{n}^* = \mathbf{r}^* \neq \{0\}$.

Vice versa, let $\mathbf{r}^*$ be vanishing. So, $\mathbf{r}(\mathfrak{q})$ is null and $\mathbf{Der}(\mathfrak{q}) = ad_{\mathfrak{q}}$. A trivial consequence is that

$$\forall x \in \mathfrak{g}, \exists! Q_x \in \tilde{\mathfrak{q}} : \psi(x) = \psi(Q_x).$$

Suppose, that x is in $\mathfrak{g}^{\perp_\psi}$. Hence, Q_x is in $\mathfrak{g}^{\perp_\psi}$, too; which means that $\mathfrak{g}^{\perp_\psi}$ is a LCR-ideal. If $\mathfrak{g}^{\perp_\psi}$ is not zero, it is a LCR-ideal on which

B^{ψ} vanishes identically. So $\mathcal{D}\mathrm{g}^{\perp_{\psi}}$ is CR-nilpotent and $\mathrm{g}^{\perp_{\psi}}$ is CR-solvable. Thus, $\mathbf{r}^* \supseteq \mathrm{g}^{\perp_{\psi}}$, that is a contradiction. So, if $\mathbf{r}^*$ vanishes, $\mathrm{g}^{\perp_{\psi}}$ vanishes. ■

Proposition 3.5.5 *If the only LCR-ideals of* g *are the trivial ones, (i.e.,* g*,* $\tilde{\mathrm{q}} = \mathrm{q} \oplus \overline{\mathrm{q}}$*, and* $\{0\}$*),* g *is CR-semisimple.*

Proof: first of all consider the case in which $\tilde{\mathrm{q}}^{\perp_{\psi}}$ is not a LCR-ideal, then $\forall Q \in \mathrm{q}$, there are $Q_1, Q_2 \in \mathrm{q}$ such that $B(Q, Q_1 + \overline{Q_2}) \neq 0$, while $B(Q, \overline{Q_2}) = 0$, so $B(Q, Q_1) \neq 0$ and $\mathrm{q}^{\perp_{\mathrm{q}}} = \{0\}$. This means that q is semisimple and hence, g is CR-semisimple.

In the case that $\tilde{\mathrm{q}}^{\perp_{\psi}}$ is a LCR-ideal, $\tilde{\mathrm{q}}^{\perp_{\psi}}$ is or $\tilde{\mathrm{q}}$ either g. In both the cases, $B|_{\tilde{\mathrm{q}}}$ vanishes identically and $\tilde{\mathrm{q}}$ is solvable. This implies that $\tilde{\mathrm{q}} \neq \mathcal{D}\tilde{\mathrm{q}}$ and any τ-stable linear subspace $\mathbf{a}$ such that $\tilde{\mathrm{q}} \supseteq \mathbf{a} \supseteq \mathcal{D}\tilde{\mathrm{q}}$ is a LCR-ideal. So $\tilde{\mathrm{q}}$ should be one-dimensional, which false. ■

Definition 3.5.6 *A LCR-algebra* g *is said to be CR-maximal if all its nontrivial LCR-ideals are contained in* $\tilde{\mathrm{q}}$*. A LCR-algebra* g *is said to be CR-simple if all its nontrivial LCR-ideals contain* $\tilde{\mathrm{q}}$*.*

Definition 3.5.7 *A chain of LCR-ideals is a family* $\mathcal{H} = \{\mathrm{h}_0 \subset \mathrm{h}_1 \subset \ldots \subset \mathrm{h}_p\}$ *such that the first element* h_1 *is not contained in* $\tilde{\mathrm{q}}$*.*

All the elements of a chain are endowed of a CR-structure of positive codimension. When the algebra is CR-semisimple, the element h_1 is CR-maximal.

3.6 CR-semisimple LCR-algebras.

In this Section we discuss the main properties of CR-semisimple LCR-algebras. Since the form B^ψ is nonsingular, for any linear subspace a, $\dim \mathrm{g} = \dim \mathrm{a} + \dim \mathrm{a}^{\perp_\psi}$. This fact is useful in the study of the LCR-ideals of such LCR-algebras.

Lemma 3.6.1 *Let* g *be a CR-semisimple LCR-algebra. If we consider a LCR-ideal* h*, we have the decompositions* $\mathrm{g} = \mathrm{h} \odot \mathrm{h}^{\perp_\psi} = (\mathrm{h} \cap \mathrm{q}) \odot (\mathrm{h} \cap \mathrm{q})^{\perp_\psi} =$*. Moreover, since* $B^\psi([x, y], z) = B^\psi(x, [y, z])$*,* $\mathrm{h}^{\perp_\psi}$ *is an ideal, whenever* h *is an ideal.*

Lemma 3.6.2 *A LCR-ideal* h *contains* q *if and only if* $\mathrm{h}^{\perp_\psi}$ *does not intersect* q.

Proof: when q is included into h, then $\mathrm{h}^{\perp_\psi}$ is contained in $\mathrm{q}^{\perp_\psi}$ which does not intersect q.
Vice versa, let $\mathrm{h}^{\perp_\psi} \cap \mathrm{q}$ vanish. Consider $K = Q + Q^\psi$ in $\mathrm{h}^{\perp_\psi}$: where Q is in q and Q^ψ is in $\mathrm{q}^{\perp_\psi}$. For any $H \in \mathrm{h}$, $[K, H] = [Q, H] + [Q^\psi, H]$ vanishes, in fact h and $\mathrm{h}^{\perp_\psi}$ are disjoint ideals. Since $[Q, H] \in \mathrm{q}$ and $[Q^\psi, H] \in \mathrm{q}^{\perp_\psi}$, then $[Q, H] = [Q^\psi, H] = 0$. In particular Q is in $\mathrm{h}^{\perp_\psi}$. Thus, Q vanishes. Hence, $\mathrm{h}^{\perp_\psi} \subseteq \mathrm{q}^{\perp_\psi}$ and $\mathrm{q} \subseteq \mathrm{h}$. ■

Corollary 3.6.3 *If* h *is a LCR-ideal, then or* h *contains* q *either* $\mathrm{h}^{\perp_\psi}$ *is a LCR-ideal.*

Theorem 3.6.4 *Let* g *be a CR-semisimple LCR-algebra and* h *be a LCR-ideal. Then*

1. $h^{\perp_\psi}$ *is a* τ*-stable ideal;*

2. *either* h *contains* q *or* $h^{\perp_\psi}$ *is a LCR-ideal;*

3. $[h, h^{\perp_\psi}] = \{0\}$;

4. h *is CR-semisimple;*

5. g/h *is CR-semisimple, whenever* h *does not contain* q.

Proof: for the first assert, take x in $h^{\perp_\psi}$. Then $B^\psi(\overline{x}, h) = B^\psi(x, h)$ vanishes, and $\overline{x} \in h^{\perp_\psi}$.
The second and the third points are given by the previous lemmas.
Let g be CR-semisimple, then q is semisimple and $h \cap q$ is a nonzero semisimple ideal of h, which means that h is CR-semisimple. Furthermore, $q/q \cap h$ is a semisimple LCR-structure of g/h. Thus g/h is CR-semisimple. ■

Corollary 3.6.5 *Let* g *be a CR-semisimple LCR-algebra and* h *be an its LCR-ideal. If* k *is a LCR-ideal (resp. an ideal) of* h*, then* k *is a LCR-ideal (resp. an ideal) of* g.

Corollary 3.6.6 *If* g *is a CR-semisimple LCR-algebra, then* g *coincides with* $\mathcal{D}g \odot \zeta(g)$.

Proof: take x in $(\mathcal{D}g)^{\perp_\psi}$ and y, z in g. Then $B^{\perp_\psi}([x, y], z) = B^{\perp_\psi}(x, [y, z]) = 0$. Thus, $(\mathcal{D}g)^{\perp_\psi}$ is contained in the centre $\zeta(g)$. Furthermore, take $[x, y]$ in $\mathcal{D}g$ and z in $\zeta(g)$, hence $B^{\perp_\psi}([x, y], z) = B^{\perp_\psi}(x, [y, z]) = 0$, and $\mathcal{D}g$ is contained in $\zeta(g)^{\perp_\psi}$. Since $\mathcal{D}g$ is a LCR-ideal, the thesis follows. ■

Theorem 3.6.7 *Let* g *be a LCR-algebra and* h *a LCR-ideal such that* g/h *is CR-semisimple, then the CR-radical is contained in* h. *Let* φ : g $\rightarrow$ g_1 *be a CR-epimorphism, then* $\varphi \mathbf{r}^* = \mathbf{r}_1^*$.

Proof: consider the canonical projection π : g $\rightarrow$ g/h and let $\mathbf{r}^*$ be not a subset of h. Then $\pi(\mathbf{r}^*)$ would be a nonzero CR-solvable LCR-ideal, which is impossible.

Since $g/\mathbf{r}^*$ is CR-semisimple, $g_1/\varphi(\mathbf{r}^*)$ is CR-semisimple. So, by the previous remark, $\varphi(\mathbf{r}^*) \supseteq \mathbf{r}_1^*$. By the other hand, $\varphi(\mathbf{r}^*)$ is a CR-solvable LCR-ideal, so $\varphi(\mathbf{r}^*) \subseteq \mathbf{r}_1^*$. ■

Theorem 3.6.8 *If* g *is a CR-semisimple LCR-algebra, then the Lie-algebra of its CR-derivations is given by* $\mathbf{Der}^*(g) = ad(g) \odot \mathbf{Der}\zeta(g)$.

Proof: since $[D, ad_Q] = ad_{DQ}$, $ad(\mathrm{q})$ is an ideal of $\mathbf{Der}^*(g)$. Obviously, $ad(\mathrm{q}) \cap \overline{ad(\mathrm{q})}$ vanishes. So, $ad(\mathrm{q})$ is a LCR-structure of $\mathbf{Der}^*(g)$.

Moreover, $ad(g)$ is CR-semisimple. In fact $ad : g \rightarrow ad(g)$ is a CR-epimorphism. Furthermore, $\mathbf{Der}^*(g)$ is CR-semisimple, too. Hence. $\mathbf{Der}^*(g)$ coincides with $ad(g) \odot (ad(g))^{\perp_\psi}$. Take, now, D in $(ad(g))^{\perp_\psi}$. then $ad_{DX} = 0$, which means that $Dg \subseteq \zeta g$. Let us define the subspaces

$$\mathcal{D}_1 \doteq \{D : Dg \subseteq \mathcal{D}g\}$$
$$\mathcal{D}_2 \doteq \{D : Dg \subseteq \zeta(g)\}.$$

Since, $ad(g)$ is in $\mathcal{D}_1$ and $(ad(g))^{\perp_\psi}$ in $\mathcal{D}_2$, then $\mathbf{Der}^*(g) = \mathcal{D}_1 + \mathcal{D}_2$. Moreover $\mathcal{D}_1 \cap \mathcal{D}_2$ vanishes, so $\mathcal{D}_1 = ad g$ and $\mathcal{D}_2 = (ad g)^{\perp_\psi}$. Take now D in $\mathcal{D}_2$, then $\mathcal{D}g \subseteq \ker D$. Thus, we identify $\mathcal{D}_2$ with $\mathbf{Der}(\zeta(g))$. ■

3.7 CR-maximal LCR-algebras.

In this Section, we study the *CR-maximal* LCR-algebras. We decompose a CR-semisimple LCR-algebra in factors, which are LCR-ideals, and consequently they are CR-semisimple (Theorem 3.7.4). Thus, we conclude with the classification of CR-maximal CR-semisimple LCR-algebras (Theorem 3.7.10).

Theorem 3.7.1 *Let* g *be a CR-maximal LCR-algebra. Then there are the three following cases:*

1. g *admits a complex structure containing* q*;*
2. q *has codimension* 1*;*
3. g *is CR-semisimple.*

Proof: remind that r^* is a LCR-ideal, then if r^* vanishes, g is CR-semisimple. When r^* coincides with g, it must be $\mathrm{g} \neq \mathcal{D}\mathrm{g}$. When $\mathcal{D}\mathrm{g}$ is not a LCR-ideal, let us consider the linear subspace $\tilde{\mathrm{q}} + \mathcal{D}\mathrm{g}$. If it is all g, then $\mathrm{h}_Q = \mathbb{C}Q + \mathbb{C}\overline{Q} + \mathcal{D}\mathrm{g}$ is a LCR-ideal, then $\mathrm{h}_Q = \mathrm{g}$ and $\mathrm{q} \cap \mathcal{D}\mathrm{g} \neq 0$, which is a contradiction. Otherwise, when $\tilde{\mathrm{q}} + \mathcal{D}\mathrm{g}$ is a proper subspace, it is a LCR-ideal and it must be $\mathcal{D}\mathrm{g} \subseteq \tilde{\mathrm{q}}$. So every τ-stable linear subspace containing $\tilde{\mathrm{q}}$ is a LCR-ideal. In this case, the codimension of q or vanishes either is 1. Finally, if $\mathcal{D}\mathrm{g}$ is a LCR-ideal, $\tilde{\mathrm{q}} + \mathcal{D}\mathrm{g}$ is it and we argue as above.

Let r^* be included in $\tilde{\mathrm{q}}$. Moreover, r^* is the radical of $\tilde{\mathrm{q}}$ and of g itself. Let us consider the two following case:

1) $\mathrm{r}^* \neq \tilde{\mathrm{q}}$; then there exists a Levi-subalgebra s of g such that $\tilde{\mathrm{q}} = \mathrm{r}^* \oplus \tilde{\mathrm{q}} \cap \mathrm{s}$ is the Levi-Mal'cev decomposition of $\tilde{\mathrm{q}}$. Let us define k as

$(\tilde{q} \cap s)^{\perp_\psi}$ and $h = r^* \oplus (k + \overline{k})$ is a LCR-ideal, which is impossible. So q is a complex structure.

2) $r^* = \tilde{q}$; consider a Levi-subalgebra s of g. When h is an ideal of s, $r^* \oplus h \oplus \overline{h}$ is a LCR-ideal and hence $g = r^* \oplus h \oplus \overline{h}$. Finally, $h \oplus q$ is a complex structure containing q. ■

If g is a CR-semisimple LCR-algebra, q is semisimple and we may write q as $q_1 \oplus \dots q_k = \sum_{i \in K} q_i$, where the q_i are simple ideals of q. So we may consider two distinct families of LCR-ideals: the first ones contain q, the second ones do not.

Remark that, if g is not CR-maximal, there exist some LCR-ideals containing q. Let h be a LCR-ideal such that $h \cap q = \sum_{i \in J} q_i$, then $h \oplus \sum_{i \in K-J} \tilde{q}_i$ is a LCR-ideal including $\tilde{q}$. Via such LCR-ideals, we give the following decomposition for g.

Proposition 3.7.2 *Let g be a CR-semisimple LCR-algebra, then we may write $g = +_{i \in I} h_i$, where the h_i are CR-maximal LCR-ideals such that $h_i \cap h_j = \tilde{q}$.*

Proof: take a LCR-ideal h of g such that $h \cap q = q$. Then h contains $\tilde{q}$. By Lemma 3.6.2, $h^{\perp_\psi}$ does not intersect $\tilde{q}$. Otherwise, $h' = \tilde{q} \oplus h^{\perp_\psi}$ is a LCR-ideal which verifies the following

$g = h + h'$

$h \cap h' = \tilde{q}$.

If one considers a LCR-ideal k of h, one gets the decomposition $g = k + k' + h'$ with the conditions $k \cap k' = k \cap h' = k' \cap h' = \tilde{q}$.

Remark that $\mathbf{k}'$ is the sum of $\tilde{\mathbf{q}}$ and of the orthogonal of $\mathbf{k}$ with respect of B^{ψ} in $\mathbf{h}$.

In this way, we construct some chains $\{\mathcal{H}_i\}_{i\in J}$ such that

$\mathbf{g} = +_{i\in J}\mathbf{h}_i$

$\mathbf{h}_i \cap \mathbf{h}_j = \tilde{\mathbf{q}}$,

where any $\mathbf{h}_i$ is the last element of its chain. Hence, it is CR-maximal.

Such a construction does not depend on the beginning LCR-ideal $\mathbf{h}$. In fact, if $\mathbf{h}_l$ were a CR-maximal LCR-ideal, with $l \notin J$, we have that $\mathbf{h}_i \cap \mathbf{h}_l$ is a LCR-ideal of $\mathbf{g}$ and there are two possible cases:

1. $\mathbf{h}_i \cap \mathbf{h}_l = \mathbf{h}_i$
2. $\mathbf{h}_i \cap \mathbf{h}_l = \tilde{\mathbf{q}}$. ■

When $\mathbf{q}$ is simple, any LCR-ideal contains $\tilde{\mathbf{q}}$ and we have the above decomposition.

Now, suppose $\mathbf{q}$ semisimple and write $\mathbf{q} = \mathbf{q}_1 \odot \ldots \odot \mathbf{q}_k$. Let us consider the sets $S_j = \{\mathbf{h}$ is a LCR-ideal : $\mathbf{h} \cap \mathbf{q} = \mathbf{q}_j\}$. Each S_j is notempty, since it contains $(\oplus_{i\neq j}\tilde{\mathbf{q}}_i)^{\perp_\psi}$.

Lemma 3.7.3 *If* $\mathbf{h}$ *is in* S_j, $\mathbf{h}^{\perp_\psi} \cap \mathbf{q} = \oplus_{i\neq j}\mathbf{q}_i$

Proof: $\mathbf{h}^{\perp_\psi} \cap \mathbf{q}$ is an ideal of $\mathbf{q}$, so it is sum of some $\mathbf{q}_i$. It is not $\mathbf{q}$, otherwise $\mathbf{h}$ would not be a LCR-ideal. Moreover, $\mathbf{q} = \mathbf{h} \cap \mathbf{q} \oplus \mathbf{h}^{\perp_\psi} \cap \mathbf{q}$. So, we have the further decomposition, given by the

Theorem 3.7.4 *If* $\mathbf{q}$ *is semisimple and it is decomposed as* $\mathbf{q} = \mathbf{q}_1 \odot \ldots \odot \mathbf{q}_k$, $\mathbf{g}$ *is decomposed as* $\mathbf{g} = \mathbf{g}_1 \odot \ldots \odot \mathbf{g}_k$, *where*

1. *each* $\mathbf{g}_i$ *is a CR-maximal LCR-ideal;*

2. $\mathfrak{g}_i \cap \mathfrak{q} = \mathfrak{q}_i$;

3. $\mathfrak{g}_i \cap \mathfrak{g}_l = \{0\}$.

Proof: let $\mathfrak{g}_1 \in S_1$ be CR-semisimple, then $\mathfrak{g}_1^{\perp_\psi}$ admits the LCR-structure $\mathfrak{q}_2 \odot \ldots \odot \mathfrak{g}_k$. By inductive hypothesis, $\mathfrak{g}_1^{\perp_\psi} = \mathfrak{g}_2 \odot \ldots \odot \mathfrak{g}_k$, where each $\mathfrak{g}_i$ is a LCR-ideal of $\mathfrak{g}_1^{\perp_\psi}$ (and hence of $\mathfrak{g}$) such that $\mathfrak{g}_i \cap \mathfrak{q} = \mathfrak{q}_i$ and $\mathfrak{g}_i \cap \mathfrak{g}_l = \{0\}$. Since $\mathfrak{g} = \mathfrak{g}_1 \odot \mathfrak{g}_1^{\perp_\psi}$, the assert is proved. ∎

Theorem 3.7.4 gives a decomposition of $\mathfrak{g}$, CR-semisimple, in CR-maximal LCR-ideals. Since each of them is CR-semisimple, in the last part of this Section, we shall describe the CR-maximal LCR-algebras which are CR-semisimple. Now on, $\mathfrak{g}$ will be a *CR-maximal CR-semisimple LCR-algebra.*

Lemma 3.7.5 *The ideal* $\tilde{\mathfrak{q}}^{\perp_\psi}$ *does not admit* τ*-stable ideals.*

Proof: let $\mathfrak{h} = \overline{\mathfrak{h}}$ be an ideal of $\tilde{\mathfrak{q}}^{\perp_\psi}$. Then, it is an ideal of $\mathfrak{g}$, so $\mathfrak{h}^{\perp_\psi}$ includes $\tilde{\mathfrak{q}}$ and it is a LCR-ideal. Since $\mathfrak{g}$ is CR-maximal, either $\mathfrak{h}^{\perp_\psi}$ is $\mathfrak{g}$ or is $\tilde{\mathfrak{q}}$. Hence, $\mathfrak{h}$ is or zero either $\tilde{\mathfrak{q}}^{\perp_\psi}$. ∎

Lemma 3.7.6 *Let* $\mathfrak{h}$ *be a nontrivial ideal of* $\tilde{\mathfrak{q}}^{\perp_\psi}$. *Then* $\tilde{\mathfrak{q}}^{\perp_\psi} = \mathfrak{h} \odot \overline{\mathfrak{h}}$.

Proof: the subspaces $\mathfrak{h} \cap \overline{\mathfrak{h}}$ and $\mathfrak{h} + \overline{\mathfrak{h}}$ are τ-stable ideals of $\tilde{\mathfrak{q}}^{\perp_\psi}$. When $\mathfrak{h} \cap \overline{\mathfrak{h}}$ is equal to $\tilde{\mathfrak{q}}^{\perp_\psi}$, $\mathfrak{h}$ coincides with $\tilde{\mathfrak{q}}^{\perp_\psi}$; when $\mathfrak{h} + \overline{\mathfrak{h}}$ vanishes, $\mathfrak{h}$ vanishes, too; in the case that $\mathfrak{h} \cap \overline{\mathfrak{h}}$ vanishes and $\mathfrak{h} + \overline{\mathfrak{h}} = \tilde{\mathfrak{q}}^{\perp_\psi}$, $\mathfrak{h}$ is a complex structure of $\tilde{\mathfrak{q}}^{\perp_\psi}$. ∎

Proposition 3.7.7 *Let* $\mathfrak{g}$ *be a CR-maximal CR-semisimple LCR-algebra. Then* $\mathfrak{q}$ *is either simple or a complex structure. In the last case,* $\mathfrak{g}$ *is semisimple.*

Proof: the ideal q may be written as $\mathrm{q} = \mathrm{q}_1 \odot \ldots \odot \mathrm{q}_k$, where the q_i are simple. In fact, it is semisimple. Then, $\mathrm{h}_1 \doteq \mathrm{q}_1 \odot \tilde{\mathrm{q}}^{\perp_\psi}$ is a LCR-ideal. If h_1 is included in $\tilde{\mathrm{q}}$, $\tilde{\mathrm{q}}^{\perp_\psi}$ vanishes and $\mathrm{g} = \tilde{\mathrm{q}}$ is semisimple. Otherwise, h_1 coincides with g. Thus $\mathrm{q} = \mathrm{q}_1$ is simple. ∎

Corollary 3.7.8 *The only LCR-ideals of a CR-maximal CR-semisimple LCR-algebra* g *are* $\{0\}$, $\tilde{\mathrm{q}}$ *and* g.

Lemma 3.7.9 *A nonvanishing ideal* h *of* $\tilde{\mathrm{q}}^{\perp_\psi}$ *does not admit ideals. Hence,* h *is or one-dimensional either simple.*

Proof: an ideal k of h is an ideal of $\tilde{\mathrm{q}}^{\perp_\psi}$. Thus, $\tilde{\mathrm{q}}^{\perp_\psi}$ coincides with $\mathrm{k} \odot \overline{\mathrm{k}}$ and k is equal to h. ∎

Now, we classify the CR-maximal CR-semisimple LCR-algebra via the "unique" ideal h of $\tilde{\mathrm{q}}^{\perp_\psi}$, which is either simple or abelian.

type	$\tilde{\mathrm{q}}^{\perp_\psi}$	g	*codim*q	g is
I	$\{0\}$	$\mathrm{q} \odot \overline{\mathrm{q}}$	0	semisimple
II	$\mathbb{C}H \oplus \mathbb{C}\overline{H}$	$\mathrm{q} \odot \overline{\mathrm{q}} \odot \mathbb{C}H \odot \mathbb{C}\overline{H}$	2	reductive
	$\mathbb{C}H = \mathbb{C}\overline{H}$	$\mathrm{q} \odot \overline{\mathrm{q}} \odot \mathbb{C}H$	1	reductive
III	$\mathrm{h} \odot \overline{\mathrm{h}}$	$\mathrm{q} \odot \overline{\mathrm{q}} \odot \mathrm{h} \odot \overline{\mathrm{h}}$	$2\dim \mathrm{h}$	semisimple
	$\mathrm{h} = \overline{\mathrm{h}}$	$\mathrm{q} \odot \overline{\mathrm{q}} \odot \mathrm{h}$	$\dim \mathrm{h}$	semisimple

Let us return to the CR-semisimple (not CR-maximal) case. Consider the decomposition in CR-maximal LCR-ideals given by Theorem 3.7.4: $\mathrm{g} = \odot_{i\in S}\mathrm{g}_i$, with $\mathrm{g}_i \cap \mathrm{q} = \mathrm{q}_i$. Then, divide S in three subsets S_1, S_2, S_3, such that i is in S_1 if and only if g_1 is of type I, and so on. Define $\mathrm{g}^I \doteq \odot_{i\in S_I}\mathrm{g}_i$, $\mathrm{g}^{II} \doteq \ldots$, $\mathrm{g}^{III} \doteq \ldots$. In particular, the above Table shows that

$$\mathrm{g}^I = \odot_{i\in S_1}\tilde{\mathrm{q}}_i$$

$$\mathrm{g}^{II} = \odot_{i\in S_2}(\tilde{\mathrm{q}}_i \odot \mathbf{C}H_i + \mathbf{C}\overline{H_i})$$

$$\mathrm{g}^{III} = \odot_{i\in S_3}(\tilde{\mathrm{q}}_i \odot \tilde{\mathrm{h}}_i).$$

We derive the following structure theorem for CR-semisimple LCR-algebras.

Theorem 3.7.10 *Let* g *be a CR-semisimple LCR-algebra. Then*

i) q *is contained in the semisimple LCR-ideal* $\mathcal{D}\mathrm{g}$*;*

ii) g *is reductive.*

Moreover, a reductive Lie-algebra admits a LCR-structure with respect of which is CR-semisimple if and only if it is noncompact. Namely, the class of all the reductive Lie-algebras is the disjoint union of two classes: the class of compact Lie-algebras and the one of CR-semisimple LCR-algebras.

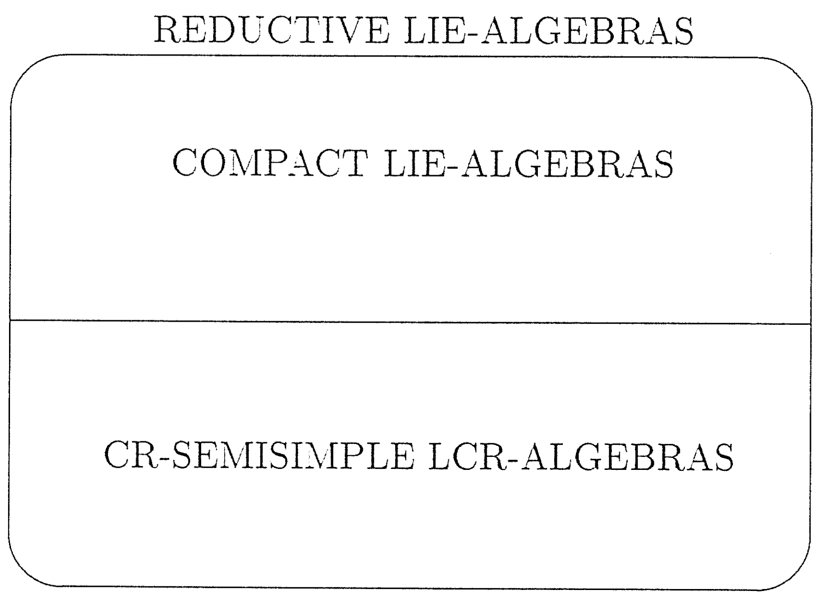
Proof: since $\tilde{\mathrm{q}}^{\perp_\psi} = \odot_{i\in S_2}(\mathbf{C}H_i \odot \mathbf{C}\overline{H_i}) \odot \odot_{i\in S_3}\tilde{\mathrm{h}}_i$, we compute

$$\mathcal{D}\tilde{\mathrm{q}}^{\perp_\psi} = \odot_{i\in S_3}\tilde{\mathrm{h}}_i$$

$$\zeta(\tilde{\mathfrak{q}}^{\perp_\psi}) = \odot_{i\in S_2}(\mathbf{C}H_i \odot \mathbf{C}\overline{H_i}).$$

Otherwise, $\mathfrak{g} = \mathcal{D}\mathfrak{g} \odot \zeta(\mathfrak{g})$. thus $\mathcal{D}\mathfrak{g} = \tilde{\mathfrak{q}} \odot \mathcal{D}\tilde{\mathfrak{q}}^{\perp_\psi}$ is a semisimple LCR-ideal and $\zeta(\mathfrak{g})$ coincides with $\zeta(\tilde{\mathfrak{q}}^{\perp_\psi})$.

Finally, when $\mathfrak{g}$ is compact $\mathcal{D}\mathfrak{g}$ is compact and semisimple, thus $\mathcal{D}\mathfrak{g}$ does not admit LCR-structures and $\mathfrak{g}$ is not CR-semisimple. Vice versa, if $\mathfrak{g}$ is a reductive noncompact Lie-algebra, $\mathcal{D}\mathfrak{g}$ admits LCR-structures, which are semisimple. ■



3.8 The CR-Levi decomposition.

This last Section is devoted to the decomposition of a LCR-algebra $\mathfrak{g}$ as the semidirect sum by *ad* of its CR-radical and a CR-semisimple LCR-algebra. In fact, there is a result analogous to Levi-Mal'cev Theorem

(Theorem 3.8.6). In order to prove this Theorem, we have to introduce the CR-cohomology of a CR-semisimple LCR-algebra.

Let g denote a CR-semisimple LCR-algebra. The element ω^{ψ} of the enveloppong algebra of g associated to the CR-polynomial $\xi^{\psi}(X) = B^{\psi}(X, X)$ is said to be the Casimir CR-element of g.

Proposition 3.8.1 *The Casimir CR-element ω^{ψ} belongs to the centre of the universal enveloping algebra of* g. *Let $\{X_i\}$ be a basis for* q, *whose dual basis is $\{X^i\}$, then $\omega^{\psi} = \sum_i X_i X^i$.*

Proposition 3.8.2 *Let ρ be a representation of* g *and* k *be $(\ker \rho)^{\perp_\psi}$. Then ω^{ρ} belongs to the centre of the universal enveloping algebra of* g. *Let $\{X_i\}$ and $\{X^i\}$ be two basis of* $\mathbf{k} \cap \mathbf{q}$ *such that $B(X_i, X^j) = \delta_i^j$, then $\omega^{\rho} = \sum_i X_i X^i$. In particular, $tr\rho(\omega^{\rho}) = \dim \mathbf{k} \cap \mathbf{q}$.*

Remark that, when ρ is nontrivial, k is a nonvanishing LCR-ideal. More generally, if $\rho(\mathbf{q}) \neq \{0\}$, then $\mathbf{k} \cap \mathbf{q} \neq \{0\}$.

Corollary 3.8.3 *Let ρ be a representation of the CR-semisimple LCR-algebra* g *and let*

$$V_n = \{v \in V : \rho(\mathbf{g})v = 0\}$$

$$V_r = +_{x \in \mathbf{g}} \rho(x)[V].$$

Then V is the direct sum of V_n and of V_r.

We now define the CR-cohomology groups: given a LCR-representation $\rho : \mathbf{g} \to \mathbf{gl}(V)$, let $V^j_{CR}(\mathbf{g}, \rho)$ be the set of the skew symmetric j-linear maps F of $g \times \ldots \times g$ (j factors) in V such that $F(q \times g \times \ldots \times g) \subseteq W$ and $F(c(q) \times g \times \ldots \times g) = 0$. If we introduce the differential operator d, obviously it maps $V^j_{CR}(g, \rho)$ in $V^{j+1}_{CR}(g, \rho)$. So, we define the CR-cohomology groups $H^j_{CR}(g, \rho)$ as the quotient $\ker d/Imd$ and we have the

Theorem 3.8.4 *If* g *is CR-semisimple and* ρ *is an its nontrivial LCR-representation, then* $H^1_{CR}(g, \rho) = H^2_{CR}(g, \rho) = \{0\}$.

The proof is analogous to the one in the semisimple case. In fact it is based on the condition $g = \mathcal{D}g \odot \zeta(g)$ and on the decomposition $C^j = C^j_n \oplus C^j_r$ (where $C^j(g, \rho) = \{\theta \in V^j(g, \rho) : d\theta = 0\}$). ■

Let, now, g be a generic LCR-algebra and r^* be its CR-radical. A *Levi sub-LCR-algebra* s^* is a sub-LCR-algebra such that $g = r^* \oplus_{ad} s^*$. This decomposition is said to be a *CR-Levi-Mal'cev decomposition.*

Lemma 3.8.5 *A Levi sub-LCR-algebra is CR-semisimple. Moreover, its centre* $\zeta(s^*)$ *vanishes*

Proof: let x be the generic element of g decomposed as $x = R_x + S_x$, and π be the natural projection on s^*. Since, there exists an element $R_Q \in q \cap r$, consider an element of q of the form $Q = R_Q + S_Q$. Hence, $S_Q \in s^* \cap q$ which is not empty. So π is a CR-epimorphism, and $r^*(s^*) = \pi(r^*) = \{0\}$.
Finally, suppose that $s^* = \zeta(s^*) \odot \mathcal{D}s^*$. Thus $r^* \odot \zeta(s^*)$ is a LCR-ideal which is CR-solvable, since $\mathcal{D}(r^* \odot \zeta(s^*)) \subseteq r^*$. Hence, $r^* = r^* \odot \zeta(s^*)$ and $\zeta(s^*)$ vanishes. ■

Theorem 3.8.6 *Any LCR-algebra* g *admits a Levi sub-LCR-algebra* s^*. *If* s^* *is a Levi sub-LCR-algebra of* g, *then it is also a Levi sub-LCR-algebra of* $\mathcal{D}\mathrm{g}$ *and the CR-Levi-Mal'cev decomposition of* $\mathcal{D}\mathrm{g}$ *is* $\mathcal{D}\mathrm{g} = [\mathrm{r}^*, \mathrm{g}] \oplus_{ad} \mathrm{s}^*$.

Proof: we make the proof by induction on $\dim \mathrm{r}^*$. If $\dim \mathrm{r}^* = 0$, g is a Levi sub-LCR-algebra. Now, let $\dim \mathrm{r}^* \geq 1$. There are two cases:

1. $\mathcal{D}\mathrm{r}^*$ is a LCR-ideal. Hence, $\mathrm{g}' = \mathrm{g}/\mathcal{D}\mathrm{r}^*$ is a LCR-algebra and $\pi(\mathrm{r}^*)$ is its CR-radical (where π is the natural projection). By the induction hypothesis, g' admits a Levi sub-LCR-algebra s'. Let us denote $\pi^{-1}(\mathrm{s}')$ as s_0. Then $\mathrm{g} = \mathrm{r}^* + \mathrm{s}_0$ and $\mathrm{q} \cap \mathcal{D}\mathrm{r}^* = \mathrm{q} \cap \mathrm{r}^* \cap \mathrm{s}_0$. Finally $\mathcal{D}\mathrm{r}^* \cap \mathrm{q}$ is a CR-solvable LCR-ideal of s_0 and $\mathrm{s}_0/\mathcal{D}\mathrm{r}^*$ is isomorphic to s'. Hence, $\mathcal{D}\mathrm{r}^*$ is the CR-radical of s_0. Since $\dim \mathcal{D}\mathrm{r}^* < \dim \mathrm{r}^*$, s_0 admits a Levi sub-LCR-algebra s such that $\mathrm{s}_0 = \mathcal{D}\mathrm{r}^* \oplus_{ad} \mathrm{s}$. Moreover $\mathrm{g} = \mathrm{r}^* \oplus_{ad} \mathrm{s}$ and s is a Levi sub-LCR-algebra of g.

2. $\mathcal{D}\mathrm{r}^*$ is not a LCR-ideal. Let us consider the subalgebra $\tilde{\mathrm{q}}$ and the Lie-epimorphism π_{q}. Hence, $\pi_{\mathrm{q}}(\mathrm{r}(\tilde{\mathrm{q}})) = \mathrm{r}(\mathrm{q})$, so $\mathrm{r}(\tilde{\mathrm{q}}) = \mathrm{r}(\mathrm{q}) \oplus \mathrm{r}(\overline{\mathrm{q}})$. Moreover, $\mathcal{D}\mathrm{r}(\tilde{\mathrm{q}}) = \mathcal{D}\mathrm{r}(q) \oplus \mathcal{D}\mathrm{r}(\overline{\mathrm{q}}) \subseteq \mathcal{D}\mathrm{r}^* \cap \mathrm{q} \oplus \mathcal{D}\mathrm{r}^* \cap \overline{\mathrm{q}}$ and $\mathrm{r}(\tilde{\mathrm{q}})$ is abelian.

The Lie-algebra $\mathrm{g}_1 = \mathrm{g}/\mathrm{r}(\tilde{\mathrm{q}})$ admits the LCR-structure $\mathrm{q}_1 = \mathrm{q}/\mathrm{r}(\mathrm{q})$ which is semisimple. Consider a linear map $\mu : \mathrm{g}_1 \to \mathrm{g}$ such that $\pi \circ \mu = id$, $\mu \mathrm{q}_1 \subseteq \mathrm{q}$ and $\mu\tau_1 = \tau\mu$. Let us define $\rho : \mathrm{g}_1 \to \mathrm{gl}(\mathrm{r}(\tilde{\mathrm{q}}))$: $X_1 \mapsto ad_{\mu(X_1)}|_{\mathrm{r}(\tilde{\mathrm{q}})}$. Since $\mathrm{r}(\tilde{\mathrm{q}})$ is abelian ρ is well defined and it is a LCR-representation.

Now, define $\theta(x, y) = [\mu x, \mu y] - \mu([x, y])$. We may easily compute that $\theta(x,y)$ belongs to $\mathrm{r}(\tilde{\mathrm{q}})$; $d\theta$ vanishes; $\theta(x,y) = 0$, when x is in $\zeta(\mathrm{g}_1)$; and $\theta(x,Q)$ belongs to $\mathrm{r}(\mathrm{q})$. These facts mean that θ is in $H^2_{CR}(\mathrm{g},\rho)$ which vanishes. So, there exists a linear map $\nu : \mathrm{g}_1 \to \mathrm{r}(\tilde{\mathrm{q}})$ which maps q_1 into q and such that $\theta = d\nu$. The map $\lambda = \mu - \nu$ is a CR-homomorphism such that $\pi \circ \lambda = id$. Hence $\mathrm{s}^* = \lambda \mathrm{g}_1$ is a sub-LCR-algebra such that $\mathrm{g} = \mathrm{r}^* \oplus_{ad} \mathrm{s}^*$.

Now, let $\mathrm{p} = [\mathrm{r}^*, \mathrm{g}]$. Then $\mathcal{D}\mathrm{g} = \mathrm{p} + [\mathrm{s}^*, \mathrm{s}^*]$. Since s^* is a Levi sub-LCR-algebra it is $\mathcal{D}\mathrm{g} = \mathrm{p} + \mathrm{s}^*$ and $\mathrm{p} \cap \mathrm{s}^* \subseteq \mathrm{r}^* \cap \mathrm{s}^* = \{0\}$. Moreover, since $\mathcal{D}\mathrm{g}$ is a LCR-ideal, $\mathrm{r}^*(\mathcal{D}\mathrm{g}) = \mathcal{D}\mathrm{g} \cap \mathrm{r}^*$. ■

Finally, the CR-version of Harish-Chandra Theorem may be proved as well as the classical one.

Theorem 3.8.7 *Let* g *be a LCR-algebra and* r^* *be its CR-radical. If* s_1^* *and* s_2^* *are Levi sub-LCR-algebras, there exists a CR-automorphism* φ *such that* $\varphi \mathrm{s}_1^* = \mathrm{s}_2^*$.

Corollary 3.8.8 *If* s^* *is a Levi sub-LCR-algebra and* s *is a CR-semi-simple sub-LCR-algebra. Then there exists a CR-automorphism* φ *such that* $\varphi \mathrm{s} \subseteq \mathrm{s}^*$.

3.9 Appendix.

1. For reasons of simplicity, we developed the structure theory of LCR-algebras in the complex terms. As we have remarked in Chapter 1. the more geometrical approach would be the real one. Thus, we translate the most interesting results about (g, q), involving $(\mathrm{g}_0, \mathrm{p}, J)$.
Remind that a real subalgebra h_0 of g_0 is a sub-LCR-algebra if it satisfies the condition

$$J(\mathrm{h}_0 \cap \mathrm{p}) \subseteq \mathrm{h}_0 \cap \mathrm{p} \neq \{0\}.$$

Define h as the complexified $\mathrm{h}_0 \otimes_{\mathbf{R}} \mathbf{C}$. Then, $\mathrm{h}_0 \cap \mathrm{p}$ vanishes if and only if $\mathrm{h} \cap \mathrm{q}$ does. Finally, a CR-homomorphism φ between two LCR-algebras $(\mathrm{g}_0, \mathrm{p}, J)$ and $(\mathrm{g}'_0, \mathrm{p}', J')$ is a Lie-homomorphism which maps p into p' and which intertwines J and J'.

2. In this terms, we give the

Proposition. *Let* h_0 *be a sub-LCR-algebra. Then the following statements are true*

1. h_0 *is CR-nilpotent if and only if* $\mathcal{C}^k\mathrm{h}_0 \cap \mathrm{p}$ *vanishes;*
2. h_0 *is CR-solvable if and only if* $\mathcal{D}^k\mathrm{h}_0 \cap \mathrm{p}$ *vanishes.*

Let us study, in particular, a CR-solvable LCR-algebra. About its LCR-structure, there is the

Theorem. *Let* (p, J) *be a LCR-structure such that* g_0 *is a CR-*

solvable LCR-algebra. Then $\mathfrak{p}$ *is contained in the radical* $\mathfrak{r}(\mathfrak{g}_0)$. *Vice versa, an evendimensional solvable ideal* $\mathfrak{p}$ *supports a unique complex structure* J *such that* $(\mathfrak{p}, J)$ *is a LCR-structure and* $\mathfrak{g}_0$ *is a CR-solvable LCR-algebra.* ∎

A characterisation of CR-solvable LCR-algebras is based on the fact that if $\mathfrak{h}_0$ is a CR-solvable LCR-ideal and $\mathfrak{g}_0/\mathfrak{h}_0$ is a CR-solvable LCR-algebra, then $\mathfrak{g}_0$ itself is CR-solvable. Two facts follow: the first consequence is that **a LCR-algebra $\mathfrak{g}_0$ is CR-solvable if and only if its derived $\mathcal{D}\mathfrak{g}_0$ is CR-nilpotent.** The second one is the existence of a maximal CR-solvable LCR-ideal $\mathfrak{r}_0^*$, said the CR-radical of $\mathfrak{g}_0$. Obviously, a CR-solvable LCR-algebra coincides with its CR-radical. Then, we define *CR-semisimple* a LCR-algebra with vanishing CR-radical.

Lemma. *Let* $\mathfrak{g}_0$ *be a LCR-algebra whose LCR-structure is* $(\mathfrak{p}, J)$. *Then, the radical* $\mathfrak{r}(\mathfrak{p})$ *of* $\mathfrak{p}$ *is given by the intersection* $\mathfrak{r}^* \cap \mathfrak{p}$ *and it is invariant under* J.

Proof: since $\mathfrak{r}^* \cap \mathfrak{p}$ is a solvable ideal of $\mathfrak{p}$, it is contained in the radical $\mathfrak{r}(\mathfrak{p}) = \mathfrak{r}(\mathfrak{g}_0) \cap \mathfrak{p}$. By Proposition 5.8 of Chapter 2, $\mathfrak{r}(\mathfrak{p})$ is invariant under J. thus, if $\mathfrak{r}(\mathfrak{p})$ is not null, $\mathfrak{r}(\mathfrak{g}_0)$ is a CR-solvable LCR-ideal. Hence, $\mathfrak{r}(\mathfrak{g}_0)$ is contained in $\mathfrak{r}^*(\mathfrak{g}_0)$ and their intersection with $\mathfrak{p}$ coincide. ∎

A direct consequence of the Lemma is that the LCR-algebra $\mathfrak{g}_0$ is CR-semisimple if and only if $\mathfrak{p}$ is semisimple. In particular, a semisimple LCR-structure $(\mathfrak{p}, J)$ is both a LCR-structure of a suitable Levi-subalgebra and a Levi-flat CR-structure of the centralizer of $\mathbf{r}$.

3. In order to introduce the Cartan's criteria, define the representation $\psi_0 : \mathrm{g}_0 \to \mathrm{gl}(\mathrm{p}) : x \mapsto ad_X|_{\mathrm{p}}$. Thus, $B^{\psi_0}(X,Y)$ is equal to $B^{\psi}(X,Y)$, for any X, Y in g_0. Hence, the criteria for CR-solvability and CR-semisimplicity are the following

1. the LCR-algebra g_0 is CR-solvable if and only if $B^{\psi_0}(X,[Y,Z])$ vanishes identically;

2. the LCR-algebra g_0 is CR-semisimple if and only if B^{ψ_0} is non-singular.

A direct consequence of the second criterion is that a CR-semisimple LCR-algebra g_0 is decomposed as $\mathrm{g}_0 = \zeta(\mathrm{g}_0) \odot \mathcal{D}\mathrm{g}_0$. In particular, we prove that g_0 is a noncompact reductive Lie-algebra. In order to do that, we define a *CR-maximal* LCR-algebra and we show that a CR-semisimple LCR-algebra is sum of CR-maximal CR-semisimple LCR-algebras.

Definition. *A LCR-algebra* $(\mathrm{g}_0, \mathrm{p}, J)$ *is said to be CR-maximal if any LCR-ideal, different from* g_0 *is contained in* p.

Notice that, if p has codimension less then 1, g_0 is a CR-maximal LCR-algebra. Vice versa, when g_0 is a CR-maximal LCR-algebra, three cases are possible:

1. g_0 admits a complex structure J_0 such that $J_0|_{\mathrm{p}} = J$;

2. p has codimension 1;

3. $\mathfrak{g}_0$ is CR-semisimple.

The third class of CR-maximal LCR-algebras takes a great importance in the structure theory of CR-semisimple LCR-algebras: let $\mathfrak{g}_0$ be a CR-semisimple LCR-algebra. Since $\mathfrak{p}$ is a semisimple ideal, there are some simple ideals $\mathfrak{p}_i$ of $\mathfrak{p}$ such that $\mathfrak{p} = \odot_{i\in K}\mathfrak{p}_i$. Moreover, each $\mathfrak{p}_i$ is J-stable. Consider, now, the set S_i of the LCR-ideal $\mathfrak{g}_i$ such that $\mathfrak{g}_i \cap \mathfrak{p} = \mathfrak{p}_i$. Then, it is possible to choice some $\mathfrak{g}_i$ such that

1. $\mathfrak{g}_i$ is a CR-maximal LCR-algebra;
2. $\mathfrak{g}_i \cap \mathfrak{g}_j = \{0\}$;

$\mathfrak{g} = \odot_{i\in K}\mathfrak{g}_i$.

Thus, via the CR-maximal LCR-ideals, we describe the whole CR-semisimple LCR-algebra. Of course, each of them is CR-semisimple, since it is a LCR-ideal.

Furthermore, the only LCR-ideals of a CR-maximal CR-semisimple LCR-algebra $(\mathfrak{g}_0, \mathfrak{p}, J)$ are the trivial ones: $\{0\}, \mathfrak{p}, \mathfrak{g}_0$. Finally, the ideal $\mathfrak{p}^{\perp_{\psi_0}}$ assumes one of the following forms;

$$\mathfrak{p}^{\perp_{\psi_0}} = \begin{cases} \{0\} \\ \mathbf{R}H \\ \mathfrak{h}_0 \end{cases}$$

So, a CR-maximal LCR-algebra is reductive and its centre either is onedimensional or vanishes. Let us return to the generic CR-semisimple LCR-algebra $(\mathfrak{g}_0, \mathfrak{p}, J)$. We conclude showing that $\mathfrak{p}$ is contained in the semisimple LCR-ideal $\mathcal{D}\mathfrak{g}_0$ and that $\mathfrak{g}_0$ is reductive.

In fact, since $\mathfrak{p}$ is semisimple, it coincides with its derived and so is included in $\mathcal{D}\mathfrak{g}_0$. Otherwise, the CR-maximal CR-semisimple LCR-ideal $\mathfrak{g}_i$ (which are factors of $\mathfrak{g}_0$) may be divided in three families

$$I = \{\mathfrak{g}_i : \mathfrak{g}_i = \mathfrak{p}_i\}$$

$$II = \{\mathfrak{g}_i : \mathfrak{g}_i = \mathfrak{p}_i \oplus \mathbb{R}H_i\}$$

$$III = \{\mathfrak{g}_i : \mathfrak{g}_i = \mathfrak{p}_i \oplus \mathfrak{h}_i\}.$$

Let $\mathfrak{g}_0^I$ denote the direct sum of the elements of I. In a similar way, we define $\mathfrak{g}_0^{II}$ and $\mathfrak{g}_0^{III}$. By construction, $\mathfrak{g}_0^I$ and $\mathfrak{g}_0^{III}$ are semisimple and $\mathfrak{g}_0^{II}$ is reductive. Thus, the whole CR-semisimple LCR-algebra $\mathfrak{g}_0 = \mathfrak{g}_0^I \odot \mathfrak{g}_0^{II} \odot \mathfrak{g}_0^{III}$ is reductive. Moreover, $\mathcal{D}\mathfrak{g}_0 = \mathfrak{p} \oplus \oplus_i \mathfrak{h}_i$ is semisimple.

Chapter 4

CR-semisimple LCR-algebras.

4.1 Introduction to Chapter 4.

A CR-semisimple LCR-algebra $\mathfrak{g}$ is a LCR-algebra whose Killing CR-form B^{ψ} is nonsingular. The existence of such a nonsingular bilinear form is the foundation of the Theorem of existence of a Cartan sub-LCR-algebra $\mathfrak{h}$. Essentially, a Cartam sub-LCR-algebra is a maximal CR-abelian sub-LCR-algebra, whose elements are semisimple. Moreover, the decomposition in CR-root spaces is given (Theorem 4.3.1). Such a decomposition implies that $\mathfrak{h}$ is a Cartan subalgebra (i.e. $\mathfrak{h}$ coincides with its own normalizer $\mathfrak{n}(\mathfrak{h})$) and it is abelian (Theorem 4.4.1). Thus an $ad_{\mathfrak{h}}$-stability result is proved. Hence, we give a decomposition of $\mathfrak{g}$ into the semidirect sum by ad of a semisimple ideal and a reductive subalgebra. In particular, when $\mathfrak{g}$ is a CR-semisimple LCR-algebra, then there exist an ideal $\mathfrak{h}$ containing $\tilde{\mathfrak{q}}$ and a subalgebra $\mathfrak{k}$ contained in $\tilde{\mathfrak{q}}$ such that $\mathfrak{g} = \mathfrak{h} \oplus_{ad} \mathfrak{k}$. Moreover, if $\mathfrak{h}$ is decomposed as $\mathfrak{h} = \tilde{\mathfrak{q}} \odot \mathfrak{h}_1 \odot \ldots \mathfrak{h}_l$, then $\mathfrak{q}_0$ coincides with $\mathfrak{h}_1 \odot \ldots \odot \mathfrak{h}_l \odot \mathfrak{k}$ (Theorem 4.4.9).

Since the roots of $\tilde{\mathrm{q}}$ and of q_0 determines completely the LCR-algebra g (Theorem 4.5.1), the Lie-product may be described with respect of the CR-roots. Thus, we have the

$[H, X_\alpha] = \alpha(H) X_\alpha$

$$[X_\alpha, X_\beta] = \begin{cases} H_\alpha & \text{if } \beta = -\alpha \\ 0 & \text{if } \alpha + \beta \notin \Delta \\ N_{\alpha,\beta} X_{\alpha+\beta} & \text{if } \alpha + \beta \in \Delta. \end{cases}$$

Via these relations, the Chapter is concluded with a Theorem of existence of a real form g_0^* of g which admits, as an ideal, a compact real form p^* of $\tilde{\mathrm{q}}$. So, we have given a bijection between the set of CR-semisimple Lie-algebras and the one of Lie-algebras which admit an even-dimensional semisimple compact ideal.

4.2 Cartan sub-LCR-algebras.

In this Chapter, g denotes a CR-semisimple LCR-algebra whose LCR-structure is q; $\tilde{\mathrm{q}}$ is the direct sum $\mathrm{q} \odot \overline{\mathrm{q}}$. For this class of LCR-algebras, the definition of the *Cartan sub-LCR-algebras* is a direct generalization of the classical one.

Definition 4.2.1 *A Cartan sub-LCR-algebra of a CR-semisimple LCR-algebra* g *is a sub-LCR-algebra* h *such that*

1. h *is a maximal CR-abelian sub-LCR-algebra in* g*;*
2. ad_H *is a semisimple map of* g, $\forall H \in \mathrm{h}$*;*
3. $\mathrm{h} \cap \mathrm{q}$ *is a Cartan subalgebra of* q.

Proposition 4.2.2 *If* h *is a Cartan sub-LCR-algebra then* $h \cap \tilde{q}$ *is a Cartan subalgebra of* $\tilde{q}$*. Vice versa, let* h *be a maximal CR-abelian sub-LCR-algebra whose elements are semisimple. Then* h *is a Cartan sub-LCR-algebra, when* $h \cap \tilde{q}$ *is a Cartan subalgebra.*

Proof: let h be a sub-LCR-algebra. Then, $h \cap q$ is a Cartan subalgebra of q if and only if $h \cap \overline{q}$ is Cartan subalgebra of $\overline{q}$. Thus. $h \cap \tilde{q} = h \cap q \odot h \cap \overline{q}$ is an abelian subalgebra of $\tilde{q}$. Let k be a Cartan subalgebra of $\tilde{q}$ containing $h \cap \tilde{q}$. Since q and $\overline{q}$ are ideals of $\tilde{q}$, the projections π_q qnd $\pi_{\overline{q}}$ are Lie-epimorphisms. So, $\pi_q k$ is an abelian subalgebra containing $h \cap q$. Hence, $\pi_q k$ and $k \cap q$ coincide. Finally, k coincides with $h \cap \tilde{q}$ and this one is a Cartan subalgebra. The vice versa has an analogous proof. ■

The Proposition 4.2.2 shows that in the Definition 4.2.1, the third statement may be substituted with

3'. $h \cap \tilde{q}$ is a Cartan subalgebra of $\tilde{q}$.

Let g be a generic Lie-algebra. Take an element x in g and denote with $\lambda_0 = 0, \lambda_1, \dots \lambda_k$ the eigenvalues of ad_x. Then, g is decomposed as $g = \sum_{i=0}^{k} g(x, \lambda_i)$, where

$$g(x, \lambda) \doteq \{y : (ad_x - \lambda I)^k y = 0, \text{for some k}\}.$$

Remark, finally, that $g(x, \lambda)$ is a subspace of $\tilde{q}$, whenever x is an element of $\tilde{q}$ and $\lambda \neq 0$.

Lemma 4.2.3 *Let* g *be a Lie-algebra, then*

$$[\mathrm{g}(H_0,\lambda).\mathrm{g}(H_0,\mu)] \subseteq \mathrm{g}(H_0,\lambda+\mu).$$

In particular, $\mathrm{h} \doteq \mathrm{g}(H_0, 0)$ *is a subalgebra,* $\forall H_0 \in \mathrm{g}$.

Remind that H_0 is said to be a *regular* element when $\dim \mathrm{g}(H_0, 0)$ is the minimum of $\dim(\mathrm{g}(X.0)$.

Lemma 4.2.4 *When* H_0 *is regular, the subalgebra* h *is nilpotent. Moreover, if* H_0 *is a real element,* h *is* τ*-stable.*

Now, the subspaces $\mathrm{g}(x, \lambda)$ will be used in the following Lemmas, to prove the

Theorem 4.2.5 *Let* g *be a CR-semisimple LCR-algebra. Then there exists a Cartan sub-LCR-algebra* h *of* g.

Since q is semisimple, any element $x \in \mathrm{g}$ is associated to a unique element $\varphi x \in \mathrm{q}$ such that ad_x and $ad_{\varphi x}$ coincide on q. The map φ is a Lie-epimorphism and its restriction to q is the identity. As well as we have constructed φ it is possible to define $\tilde{\varphi}$ with respect to $\tilde{\mathrm{q}}$.

Let us recall a classical

Proposition 4.2.6 *Let* $f : \mathrm{g} \to \mathrm{g}'$ *be a Lie-epimorphism. Then* f *sends regular elements of* g *in regular elements of* g′ *and the rank of* g *is greater or equal to the one of* g′. *[BO2]* ■

Corollary 4.2.7 *The epimorphism* $\tilde{\varphi}$ *maps* g_{0r} *in* $\tilde{\mathrm{q}}_r$.

Lemma 4.2.8 *Let H_0 be in g_{0r}, then $h \cap \tilde{q}$ is a Cartan subalgebra for $\tilde{q}$. In particular h is a sub-LCR-algebra.*

In fact $g(H_0, 0) \cap \tilde{q}$ coincides with $\{x \in \tilde{q} : ad_x^k H_0 = 0\}$. But, when x is in $\tilde{q}$, $ad_x^k H_0 = ad_x^k \tilde{\varphi} H_0$ and $\tilde{\varphi} H_0$ is in $\tilde{q}_r$. Hence, $g(H_0, 0) \cap \tilde{q} = \tilde{q}(\tilde{\varphi} H_0, 0)$, which is a Cartan subalgebra of $\tilde{q}$. ■

The Lie-epimorphism $\tilde{\varphi}$ maps h onto $h \cap \tilde{q}$. In fact, the proof of Lemma 4.2.8 shows that $\tilde{\varphi} h$ is included in $h \cap \tilde{q}$. While $h \cap \tilde{q} = \tilde{\varphi}(h \cap \tilde{q}) \subseteq \tilde{\varphi} h$.

Lemma 4.2.9 *Let g be a CR-semisimple LCR-algebra, then h is CR-abelian.*

Proof: take $x \in g(H_0, \lambda)$ and $y \in h$. Then $ad_x ad_y$ maps $g(H_0, \mu)$ in $g(H_0, \lambda + \mu)$: so its trace vanishes. Otherwise, since h is nilpotent, $B([H_1, H_2], H_3)$ vanishes, if each H_i is in h. Thus, $\mathcal{D}h$ is contained in $g_{\perp}$.
Finally, let $[H_1, H_2]$ be in $\mathcal{D}h \cap q$, then

$$B^{\psi}([H_1, H_2], x) = B([H_1, H_2], x) = 0,$$

for all x in g, and $[H_1, H_2]$ is in $g^{\perp_\psi}$ which vanishes. ■

Lemma 4.2.10 *Let H_0 be in g_{0r}, then h is maximal CR-abelian.*

Proof: suppose there exists a CR-abelian sub-LCR-algebra k containing h. Since $h \cap \tilde{q}$ is a Cartan subalgebra, $k \cap \tilde{q}$ coincides with $h \cap \tilde{q}$. Take $k_v \supseteq h_v$ such that $k = k \cap \tilde{q} \oplus k_v$ and $h = h \cap \tilde{q} \oplus h_v$. Consider a linear subspace l such that $\mathrm{k}_v = \mathrm{h}_v \oplus \mathrm{l}$. Then $\mathrm{k} = \mathrm{h} \oplus \mathrm{l}$. Trivially, $ad_{H_0}|_{\mathrm{l}}$ is invertible. Consider an element $L \in \mathrm{l}$, then $ad_{H_0}L \in \mathrm{h}$ and there is an integer k such that $ad^k_{H_0}L = 0$. So L vanishes. ■

Lemma 4.2.11 *Let H_0 be in g_{0r}, then ad_x is a semisimple map of g, for any x in h.*

Proof: let us consider the decomposition $\mathrm{g} = \sum_\lambda \mathrm{g}(H_0, \lambda)$ and the linear subspace $V_\beta = \{x : (ad_H - \beta(H)I)^k x = 0, \forall H \in \mathrm{h}\}$. Obviously, when $\beta(H_0)$ is equal to λ, V_β is included in $\mathrm{g}(H_0, \lambda)$. Hence, there exist some β_i such that $\mathrm{g} = \sum_i V_{\beta_i}$.

Take a generic element $H \in \mathrm{h}$. Then, there is given the canonical decomposition $ad_H = S + N$, where S is a semisimple derivation and N is a nilpotent one. Since S is a polinomial in ad_H, $S\mathrm{h}$ is contained in h. In a previous Lemma, we have shown that $ad_H|_{\mathrm{h}}$ is nilpotent; so, $S|_{\mathrm{h}}$ vanishes identically. Moreover, S is a derivation of g and there is an element Z in the centralizer of h, such that $S = ad_Z$.

Let us define the sub-LCR-algebras $\mathrm{h}_Z \doteq \mathrm{h} \oplus \mathbf{C}ReZ$ and $\mathrm{h}'_Z \doteq \mathrm{h} \oplus \mathbf{C}ImZ$. Since, $\mathcal{D}\mathrm{h}_Z = \mathcal{D}\mathrm{h}'_Z = \mathcal{D}\mathrm{h}$, they are CR-abelian and Z is in h.

Furthermore, S maps each V_β in itself and $SX = \beta(H)X$, for all X in V_β. Take, now, an eigenvector $X' \in V_\beta$. Then $SX' = \beta(Z)X'$ and $\beta(H) = \beta(Z)$. Hence, $B(H, H') = \sum_i \beta_i(H)\beta_i(H') dim V_{\beta_i}$.

A direct computation shows that the subspace $\tilde{\mathrm{q}} \cap \mathrm{g}(H_0, \lambda)$ coincide with $\tilde{\mathrm{q}}(\tilde{\varphi} H_0, \lambda)$; while the map $\psi(H) = ad_H|_{\tilde{\mathrm{q}}}$ maps $V_\beta \cap \tilde{\mathrm{q}}$ in itself. Then, it is

$$B^{\psi}(H,H') = \sum_i \beta_i(H)\beta_i(H') dim V_{\beta_i} \cap \tilde{q}.$$

So $B^{\psi}(Z-H,H')$ vanishes. Thus, since $B^{\psi}(Z-H,x)=0$, for all $x \in g(H_0,\lambda)$, with $\lambda \neq 0$, it follows that $H=Z$. ∎

The existence of a Cartan sub-LCR-algebra will be used, in the next Section, to decompose the CR-semisimple LCR-algebra g in its CR-root spaces. This decomposition will show directely the existence of a real form g_0^* which admits a compact ideal p^* which is a real form of $\tilde{q}$ (Theorem 4.5.4).

4.3 CR-root space decomposition.

Following the classical structure theory of semisimple Lie-algebras, [HE], and via the existence of a Cartan Sub-LCR-algebra h, we study the structure theory of CR-semisimple LCR-algebras.

Let α be a linear function on the complex vector space h. With g^{α} we shall denote the linear subspace of g,

$$g^{\alpha} \doteq \{x \in g : [H,x] = \alpha(H)x, \forall H \in h\}.$$

When g^{α} does not vanish, α is said to be a *CR-root*. In that case g^{α} is a *CR-root space*. Obviously, g^0 coincides with h and $[g^{\alpha}, g^{\beta}] \subseteq g^{\alpha+\beta}$, as a consequence of the Jacobi identity. The set of CR-roots is denoted by Δ. In the terms of these notations, we give the

Theorem 4.3.1 *Let* h *be a Cartan sub-LCR-algebra of* g*. Let* Δ *and* $\tilde{\Delta}$ *denote the set of CR-roots of* g *and the set of roots of* $\tilde{\mathrm{q}}$*, respectively. The following statements are true:*

(i) $\mathrm{g} = \mathrm{h} \oplus \oplus_{\alpha\in\Delta}\mathrm{g}^{\alpha}$.

(ii) the CR-root spaces g^{α} *and* g^{β} *are orthogonal under* B*, whenever* $\alpha + \beta \neq 0$.

(iii) the restriction of B^{ψ} *to* $\mathrm{h}\times\mathrm{h}$ *is nonsingular. For each linear form* α *on* h *there exists a unique element* $H_\alpha \in \mathrm{h}$ *such that* $B^{\psi}(H, H_\alpha) = \alpha(H)$*, for all* $H \in \mathrm{h}$.

(iv) if $\alpha \in \Delta$*, then* $-\alpha \in \Delta$*,* $[\mathrm{g}^{\alpha}, \mathrm{g}^{-\alpha}] = \mathbf{C}H_\alpha$ *and* $\alpha(H_\alpha) \neq 0$.

(v) $dim\,\mathrm{g}^{\alpha} = 1$.

Proof: (i) if the subspaces h and g^{α}, $\alpha \in \Delta$, were linearly dependents, there would be some $H \in \mathrm{h}$ and $X_\alpha \in \mathrm{g}^{\alpha}$ such that $0 = H + \sum_\alpha X_\alpha$. Choose H_1 in h such that $\alpha(H_1) \neq 0$, for all $\alpha \in \Delta$. Then,

$$0 = [H_1, H] + \sum_\alpha [H_1, X_\alpha] = [H_1, H] + \sum_\alpha \alpha(H_1) X_\alpha.$$

Hence, $[H_1, H]$ and $\alpha(H_1)$ vanish: so there is a contradiction. Thus, the sum $\mathrm{h} \oplus \oplus_{\alpha\in\Delta}\mathrm{g}^{\alpha}$ is direct.

Obviously, $[\mathrm{h}\cap\tilde{\mathrm{q}}, \mathrm{h}] \subseteq \mathrm{h}$ and $[\mathrm{h}\cap\tilde{\mathrm{q}}, \mathrm{g}^{\alpha}] \subseteq \mathrm{g}^{\alpha}$. Furthermore, $ad_{\mathrm{g}}(\mathrm{h}\cap\tilde{\mathrm{q}})$ is an abelian family of semisimple elements, so it is semisimple. In this hypothesis there exist some one-dimensional invariant subspaces g_i such that $\mathrm{g} = \sum_i \mathrm{g}_i$; whenever, for any i there exists an α such that $\mathrm{g}_i \subseteq \mathrm{g}^{\alpha}$. This fact concludes the proof of (i).

(ii) when X is in g^α and Y is in g^β, $ad_X ad_Y$ maps h in $\mathrm{g}^{\alpha+\beta}$ and g^γ in $\mathrm{g}^{\alpha+\beta+\gamma}$. In particular, its trace vanishes.

(iii) let H_0 be such that $B^\psi(H_0, H) = 0$, for all H in h. Consider the generic element of g, $X = H + \sum_\alpha X_\alpha$. Then, it is $B^\psi(H_0, X) = \sum_\alpha B^\psi(H_0, X_\alpha)$. Let us compute the trace of $\psi(H_0)\psi(X_\alpha) : \tilde{\mathrm{q}} \to \tilde{\mathrm{q}}$. Remind that, since $\tilde{\mathrm{q}}$ is semisimple, it is decomposed as

$$\tilde{\mathrm{q}} = \mathrm{h} \cap \tilde{\mathrm{q}} \oplus \oplus_{\tilde{\alpha} \in \tilde{\Delta}} \tilde{\mathrm{q}}^{\tilde{\alpha}}.$$

Consider, now, the map $j : \tilde{\Delta} \to \Delta : \tilde{\alpha} \mapsto \tilde{\alpha} \circ \tilde{\varphi}$. Since $\tilde{\varphi}|_{\tilde{\mathrm{q}}}$ is the identity, $\mathrm{g}^{j\tilde{\alpha}} \supseteq \tilde{\mathrm{q}}^{\tilde{\alpha}}$. Hence, $j\tilde{\alpha}$ is a CR-root of g. By direct calculation, we show the following inclusions

$$\psi(H_0)\psi(X_\alpha) \left\{ \begin{array}{l} \mathrm{h} \cap \tilde{\mathrm{q}} \subseteq \mathrm{g}^\alpha \\ \tilde{\mathrm{q}}^{\tilde{\alpha}} = \{0\} \\ \tilde{\mathrm{q}}^{\tilde{\alpha}} \subseteq \mathrm{g}^{\alpha + j\tilde{\alpha}}. \end{array} \right.$$

Remark that $\tilde{\mathrm{q}}^{\tilde{\alpha}} \cap \mathrm{g}^{\alpha+j\tilde{\alpha}} \subseteq \mathrm{g}^{j\tilde{\alpha}} \cap \mathrm{g}^{\alpha+j\tilde{\alpha}} = \{0\}$. So, the trace of $\psi(H_0)\psi(X_\alpha)$ vanishes and H_0 must be zero, since B^ψ is nondegenerate.

(iv) let X_α be in g^α, while $\mathrm{g}^{-\alpha}$ vanishes. Then $B^\psi(X_\alpha, X)$ should vanish, for all $X \in \mathrm{g}$, which is false. Now, compute

$$B^\psi([X_\alpha, X_{-\alpha}], H) = B^\psi(X_\alpha, [X_{-\alpha}, H]) = B^\psi(X_\alpha, X_{-\alpha}) B^\psi(H_\alpha, H).$$

Hence, $[X_\alpha, X_{-\alpha}] = B^\psi(X_\alpha, X_{-\alpha}) H_\alpha$. Finally $\alpha(H_\alpha) = B^\psi(H_\alpha, H_\alpha) \neq 0$. And ($iv$) is proved.

The proof of (v) is the same as in the semisimple case, cf. [HE]. ■

Corollary 4.3.2 *The map* $j : \tilde{\Delta} \to \Delta$ *is injective.*

In fact, since g^{α} is one-dimensional, $\mathrm{g}^{j\tilde{\alpha}} = \tilde{\mathrm{q}}^{\tilde{\alpha}}$. Now, let us divide the set of the CR-roots as follows: $\Delta = \Delta_0 \cup \Delta_1$, where $\Delta_0 \doteq \{\alpha : \mathrm{h} \cap \tilde{\mathrm{q}} \subseteq Ker\alpha\}$ and Δ_1 is its complement. It is not difficult to see that the map $j_1 : \Delta_1 \to \tilde{\Delta} : \alpha \mapsto \alpha|_{\mathrm{h}\cap\tilde{\mathrm{q}}}$ is injective; and that $\tilde{\mathrm{q}}^{j_1\alpha} = \mathrm{g}^{\alpha}$. Furthermore, there is the

Proposition 4.3.3 *The sets Δ_1 and $\tilde{\Delta}$ have the same cardinality. Moreover $j_1 \circ j$ (resp $j \circ j_1$) is the identity of $\tilde{\Delta}$ (resp. Δ_1).*

Proof: an easy computation shows that

$$j_1 \circ j\tilde{\alpha} = j_1\tilde{\alpha} \circ \tilde{\varphi} = \tilde{\alpha} \circ \tilde{\varphi}|_{\tilde{\mathrm{q}}} = \tilde{\alpha}$$

$$\mathrm{g}^{j \circ j_1 \alpha} = \tilde{\mathrm{q}}^{j_1\alpha} = \mathrm{g}^{\alpha} \quad \blacksquare$$

The following Proposition 4.3.4 and Lemma 4.3.5 will be useful to give a decomposition in real subalgebras of the Cartan sub-LCR-algebra h.

Proposition 4.3.4 *Let α be in Δ and β be any CR-root. Define the α-series containing β as the set of all roots of the form $\beta + n\alpha$ where n is an integer. Then*

(i) the α-series containing β is an uninterrupted string of the form $\beta + n\alpha$ $(p \leq n \leq q)$. The integers p and q satisfy the condition

$$-2\frac{\beta(H_\alpha)}{\alpha(H_\alpha)} = p + q.$$

(ii) let X_α be in $\mathrm{g}^{\alpha}, X_{-\alpha}$ in $\mathrm{g}^{-\alpha}$, and X_β in g^{β}. Then,

$$[X_{-\alpha},[X_\alpha,X_\beta]] = \frac{q(1-p)}{2}\alpha(H_\alpha)B^\psi(X_\alpha,X_{-\alpha})X_\beta.$$

(iii) the only roots proportional to α are $-\alpha, 0, \alpha$.

(iv) suppose $\alpha+\beta \neq 0$. Then, $[\mathrm{g}^\alpha,\mathrm{g}^\beta]=\mathrm{g}^{\alpha+\beta}$. ∎

Since the Killing CR-form is nonvanishing, it is possible to consider a family of elements $\{E_\alpha \in \mathrm{g}^\alpha\}_{\alpha\in\Delta}$ such that $B^\psi(E_\alpha,E_{-\alpha})=1$. This fact is the foundation of the proof of Proposition 4.3.4. The complete computations coincide with the ones of the semisimple case (in which the Killing form B replaces the CR-one B^ψ) as developed in [HE].

Lemma 4.3.5 *An element $H\in\mathrm{h}$ such that $\alpha(H)=0$, for all $\alpha\in\Delta_1$, is in the centralizer $c(\tilde{\mathrm{q}})$.*

In fact, since any CR-root α of Δ_1 is of the form $\tilde\alpha\circ\tilde\varphi$, with $\tilde\alpha$ in $\tilde\Delta$, then $\tilde\varphi H$ vanishes. ∎

A direct consequence of Proposition 4.3.4 is that on the real subspace $\mathrm{h}_{\mathbf{R}} \doteq \sum_{\alpha\in\Delta}\mathbf{R}H_\alpha$, the Killing CR-form B^ψ is real and positive definite. Moreover, $\mathrm{h}_{\mathbf{R}}$ is a real form of h: $\mathrm{h}=\mathrm{h}_{\mathbf{R}}\oplus i\mathrm{h}_{\mathbf{R}}$.
In the last part of the present Section we shall prove that both $\tilde{\mathrm{q}}$ and q may be seen as sums of their intersection with h and some CR-root spaces.

Lemma 4.3.6 *Let α be a CR-root in Δ_1. Then $H_\alpha=\tilde H_{j_1\alpha}$, where H_α and $\tilde H_{\tilde\alpha}$ are defined by*

$$B^{\psi}(H_{\alpha}, H) = \alpha(H), \forall H \in \mathrm{h};$$

$$B_{\tilde{\mathrm{q}}}(\tilde{H}_{\tilde{\alpha}}, H) = \tilde{\alpha}(H), \forall H \in \mathrm{h} \cap \tilde{\mathrm{q}}.$$

Proof: a direct computation shows that, if $\tilde{\alpha} = j_1\alpha$,

$$B^{\psi}(\tilde{H}_{\tilde{\alpha}}, H) = B_{\tilde{\mathrm{q}}}(\tilde{H}_{\tilde{\alpha}}, \tilde{\varphi}H) = \tilde{\alpha}\tilde{\varphi}(H) = \alpha(H) = B^{\psi}(H_{\alpha}, H) \blacksquare$$

Finally, recall the following notations: let Γ be a subset of Δ. We denote with h_{Γ} the subspace $\sum_{\alpha\in\Gamma} \mathbf{C}H_{\alpha}$ and with g^{Γ}, the subspace $\oplus_{\alpha\in\Gamma}\mathrm{g}^{\alpha}$. Remark that $[\mathrm{h}, \mathrm{g}^{\Gamma}] \subseteq \mathrm{g}^{\Gamma}$ and $[\mathrm{g}^{\Gamma}, \mathrm{g}^{\Gamma_1}] \subseteq \mathrm{g}^{(\Gamma+\Gamma_1)\cap\Delta} \oplus \mathrm{h}_{\Gamma\cap(-\Gamma_1)}$. In particular we shall write h_j and g^j for the subspaces h_{Δ_j} and g^{Δ_j}, respectively. In these terms, Lemma 4.3.6 says that $\mathrm{q} = \mathrm{h}_1 \oplus \mathrm{g}^1$.

Let α_j be in Δ_j. By definition of Δ_0, $\alpha_0(H_{\alpha_1})$ vanishes. In fact H_{α_1} is in $\mathrm{h} \cap \tilde{\mathrm{q}}$. This means that $B(H_{\alpha_0}, H_{\alpha_1}) = 0$. In particular, there is the

Proposition 4.3.7 *The bilinear forms $B^{\psi}|_{\mathrm{h}_0\times\mathrm{h}_0}$ and $B^{\psi}|_{\mathrm{h}_1\times\mathrm{h}_1}$ are nonsingular. Moreover, $\mathrm{h}_1 = \cap_{\alpha_0\in\Delta_0} Ker\alpha_0$ and $\mathrm{h}_0 = \cap_{\alpha_1\in\Delta_1} Ker\alpha_1$.*

Proof: the first part is a consequence of the fact that $B^{\psi}|_{\mathrm{h}\times\mathrm{h}}$ is nonsingular. Then, the above computations show that h_{Δ_1} is a subset of $\cap_{\alpha_0\in\Delta_0} Ker\alpha_0$. Finally, take, $H = h^{\alpha_1}H_{\alpha_1} + h^{\alpha_0}H_{\alpha_0}$ in $\cap_{\alpha_0\in\Delta_0} Ker\alpha_0$. By definition, it is $\beta_0(H) = h^{\alpha_0}\beta_0(H_{\alpha_0})$. Decompose h^{α_0} as $a^{\alpha_0} + ib^{\alpha_0}$ and define $A \doteq a^{\alpha_0}H_{\alpha_0}$ and $B \doteq b^{\alpha_0}H_{\alpha_0}$. Then $B(H_{\beta}, A) = B(H_{\beta}, B) = 0$, $\forall\beta \in \Delta$. Thus A and B vanish. $\blacksquare$

Let us recall that when $\mathfrak{h}'$ is a subspace of $\mathfrak{h}$ and Γ is a subset of Δ, then the linear space $\mathfrak{h}' \oplus \mathfrak{g}^{\Gamma}$ is a subalgebra if and only if Γ is closed and $\mathfrak{h}' \supseteq \mathfrak{h}_{\Gamma\cap(-\Gamma)}$.

Define, now, the subsets

$$\Delta_1(\mathfrak{q}) \doteq \{\alpha \in \Delta_1 : Ker\alpha \text{ does not contain } \mathfrak{h}\cap\mathfrak{q}\}$$

$$\Delta_1(\overline{\mathfrak{q}}) \doteq \{\alpha \in \Delta_1 : Ker\alpha \text{ does not contain } \mathfrak{h}\cap\overline{\mathfrak{q}}\}.$$

Since $\mathfrak{q}$ and $\overline{\mathfrak{q}}$ are ideals of the semisimple Lie-algebra $\tilde{\mathfrak{q}}$, they are $ad_{\mathfrak{h}\cap\tilde{\mathfrak{q}}}$-stable. Hence, we may apply the

Lemma 4.3.8 *Let* $\mathfrak{h}$ *be a Cartan subalgebra of a semisimple Lie-algebra* $\mathfrak{g}$ *and* V *a linear subspace of* $\mathfrak{g}$. *Define the set* $\Delta(V)$ *of the roots* $\alpha \in \Delta$ *such that* $\mathfrak{g}^{\alpha} \subseteq V$. *Then the greatest linear subspace of* V *which is* $ad_{\mathfrak{h}}$*-stable is* $V \cap \mathfrak{h} + \mathfrak{g}^{\Delta(V)}$. *[BO2].* ■

So, we obtain that $\mathfrak{q} = \mathfrak{h}\cap\mathfrak{q} \oplus \mathfrak{g}^{\Delta_1(\mathfrak{q})}$ Since, $\tilde{\mathfrak{q}} = \mathfrak{h}\cap\mathfrak{q} \oplus \mathfrak{h}\cap\overline{\mathfrak{q}} \oplus \mathfrak{g}^{1}$, the following relations are true:

(i) $\mathfrak{h}\cap\tilde{\mathfrak{q}} = \mathfrak{h}\cap\mathfrak{q} \oplus \mathfrak{h}\cap\overline{\mathfrak{q}}$;

(ii) $\Delta_1 = \Delta_1(\mathfrak{q}) \cup \Delta_1(\overline{\mathfrak{q}})$.

In particular, when α is in $\Delta_1(\mathfrak{q})$, $-\alpha$ is in it, too. And the Cartan subalgebra $\mathfrak{h}\cap\mathfrak{q}$ coincides with $\mathfrak{h}_{\Delta_1(\mathfrak{q})}$. ■

Remark 4.3.9 *The above decomposition gives a construction for different CR-structures of* $\mathfrak{g}$. *Let* $\Delta^* \subseteq \Delta$ *be a closed subset such that*

1. $\Delta^* \cap \overline{\Delta}^* = \{0\}$

2. $[H_\alpha, H_\beta] = 0, \forall \alpha, \beta \in \Delta^*$.

Then, the subspace $q^* \doteq h_{\Delta^*} \oplus g^{\Delta^*}$ *is a CR-structure.*

Proposition 4.3.10 *The closed set* $\Delta^\alpha = \{\pm\alpha, 0\}$ *satisfies the two conditions of Remark 4.3.9.*

In fact, the first one is trivial. For the second, let us compute

$$\begin{aligned} B^\psi([H_\alpha, H_{-\alpha}], H) &= B^\psi(H, [H_\alpha, H_{-\alpha}]) = B^\psi([H, H_\alpha], H_{-\alpha}) = \\ &= B^\psi([H_\alpha, H], H_\alpha) = B^\psi(H_\alpha, [H_\alpha, H]) = \\ &= B^\psi([H_\alpha, H_\alpha], H) = 0. \ \blacksquare \end{aligned}$$

4.4 A decomposition of g.

Recall that a Cartan subalgebra **h** is a nilpotent subalgebra which coincides with its normalizer n(**h**). In this Section, we proof that a Cartan sub-LCR-algebra is an abelian Cartan subalgebra. Hence, we make use of the abelianity to decompose the CR-semisimple LCR-algebra g. The final result is based on some facts about ad_{h}-stability proved in [BO2].

Theorem 4.4.1 *Let* h *be a Cartan sub-LCR-algebra. Then* h *is a Cartan subalgebra of* g.

Proof: by Lemma 3.2.6, the subalgebra **h** is nilpotent. Moreover, take an element $X = H + \sum_{\alpha\in\Delta} X_\alpha$ of n(**h**). Then, by definition $[X, H'] = [H, H'] + \sum_{\alpha\in\Delta} \alpha(H')X_\alpha$ is in **h**, for all $H' \in$ **h**. Hence, X_α vanishes, for all α in Δ. ■

Even the converse is true. In fact,

Proposition 4.4.2 *Let* h *be a* τ*-stable Cartan subalgebra of* g *such that* $\mathrm{h} \cap \mathrm{q}$ *is a Cartan subalgebra of* q. *Then* h *is a Cartan sub-LCR-algebra of* g

Proof: $ad_H \mathrm{g} \to \mathrm{g}$ is a semisimple map and h is a CR-abelian sub-LCR-algebra. The maximality of h is shown as in Lemma 4.2.10. ■

Proposition 4.4.3 *Let* h *be a Cartan subalgebra of* g *which is a sub-LCR-algebra. then* h *is a Cartan sub-LCR-algebra if and only if* $\mathrm{h} \cap \mathrm{q}$ *is a Cartan subalgebra of* q. ■

Moreover, h has the same properties as the Cartan subalgebra of a semisimple Lie-algebra.

Proposition 4.4.4 *The Cartan subalgebra* h *is a maximal abelian subalgebra of* g.

Proof: let us compute
$B^{\psi}([H_1, H_2], H_3) = B_{\tilde{\mathrm{q}}}(\tilde{\varphi}[H_1, H_2], \tilde{\varphi}H_3) = B_{\tilde{\mathrm{q}}}([\tilde{\varphi}H_1, \tilde{\varphi}H_2], \tilde{\varphi}H_3)$.
Finally, $[\tilde{\varphi}H_1, \tilde{\varphi}H_2]$ vanishes, by the abelianity of $\mathrm{h} \cap \tilde{\mathrm{q}}$. Hence, since B^{ψ} is nondegenerate on $\mathrm{h} \times \mathrm{h}$, $[H_1, H_2]$ vanishes, too. The maximality follows by the definition. ■

Since h is abelian, the ad_{h}-stable linear subspaces are described by the

Lemma 4.4.5 *Let* V *be a linear subspace of* g *and* $\Delta(V)$ *the set* $\{\alpha \in \Delta : \mathrm{g}^{\alpha} \subseteq V\}$. *Then, the greatest* ad_{h}*-stable linear subspace of* V *is* $V \cap \mathrm{h} + \mathrm{g}^{\Delta(V)}$. ■

As a consequence of Lemma 4.4.5, we describe the ad_{h}-stable subalgebras.

Proposition 4.4.6 *The ad_{h}-stable subalgebras of* g *are the linear subspaces* $\mathrm{h}' \oplus \mathrm{g}^{\Gamma}$*, where* $\Gamma \subseteq \Delta$ *is a closed subset and* $\mathrm{h}' \subseteq \mathrm{h}$ *is a linear subspace including* $\mathrm{h}_{\Gamma\cap(-\Gamma)}$. ■

Proposition 4.4.7 *Let* $\mathrm{k} \subseteq \mathrm{g}$ *be an ad_{h}-stable subalgebra,* h' *a subspace of* h *and* Γ *a subset of* Δ *such that* $\mathrm{k} = \mathrm{h}' \oplus \mathrm{g}^{\Gamma}$*. Then,* k *is reductive if and only if* $\Gamma = -\Gamma$. ■

Now, we have all the elements to give the main result of the Section. In the previous Section we have decomposed g as $\mathrm{g} = \tilde{\mathrm{q}} \oplus \mathrm{h}_0 \oplus \mathrm{g}^0$. Let us pose $\mathrm{q}_0 \doteq \mathrm{h}_0 \oplus \mathrm{g}^0$. Since Δ_0 is a closed set such that $\Delta_0 = -\Delta_0$, q_0 is an ad_{h}-stable complex subalgebra of g. Moreover $\mathrm{h}_1 \subseteq c(\mathrm{q}_0)$ and $\mathrm{q}_0 \subseteq \mathrm{n}(\mathrm{g}^1)$. Finally, remark that $\mathrm{g} = \tilde{\mathrm{q}} \oplus_{ad} \mathrm{q}_0$, and we have proved the

Theorem 4.4.8 *Let* g *be CR-semisimple. Then, there exists a reductive subalgebra* q_0 *such that* $\mathrm{g} = \tilde{\mathrm{q}} \oplus_{ad} \mathrm{q}_0$*. The subalgebra* h_0 *is a Cartan subalgebra of* q_0. ■

To give a deeper description of $\mathrm{g} = \tilde{\mathrm{q}} \oplus_{ad} \mathrm{q}_0$, let us study a Lie-algebra g decomposed as $\mathrm{g} = \mathrm{h} \oplus_{\delta} \mathrm{k}$, where the first factor is semisimple and the second is reductive.
As we have remarked in Chapter 1, since h is semisimple, there exists a Lie-homomorphism $B : \mathrm{k} \to \mathrm{h}$ such that $\delta(K) = ad_{BK}$, $\forall K \in \mathrm{k}$.

Consider now the decompositions in simple ideal $h = h_1 \odot \ldots \odot h_h$ and $k = k_0 \odot k_1 \odot \ldots \odot k_k$, where k_0 is the centre $\zeta(k)$. Thus, via a permutation, the ideal $KerB$ may be seen as $KerB = k_{\beta_0} \odot \ldots \odot k_{\beta_b}$ and $k = KerB \odot k_{\beta_{b+1}} \odot \ldots \odot k_{\beta_k}$. Remind that, when $KerB$ coincides with k, δ vanishes and the sum is direct.

Moreover, define $h^B = h \odot KerB$ and $k^B = k_{\beta_{b+1}} \odot \ldots \odot k_{\beta_k}$. Then h^B is an ideal of g, k^B is an its subalgebra and one of them is semisimple. Furthermore, the map $\hat{B} : k^B \to h : K \mapsto B(K)$ is injective and k^B is isomorphic to the subalgebra Bk^B of h. Finally, remark that the following decompositions of g are given

$$g = h \oplus_\delta k = h^B \oplus_\delta k^B \simeq h^B \oplus_{ad} Bk^B.$$

Theorem 4.4.9 *Let* g *be a CR-semisimple not semisimple LCR-algebra. Then there exist an ideal* h *containing* $\tilde{q}$ *and a subalgebra* k *contained in* $\tilde{q}$ *such that* $g = h \oplus_{ad} k$. *Moreover, if* h *is decomposed as* $h = \tilde{q} \odot h_1 \odot \ldots h_l$, *then* q_0 *coincides with* $h_1 \odot \ldots \odot h_l \odot k$. ■

4.5 Real CR-forms.

Let g and g′ be two Lie-algebras endowed with two semisimple LCR-structures q and q′. Since $\tilde{q}$ and $\tilde{q}'$ are semisimple Lie-algebras, any one-to-one $\mathbf{R}$-linear map $f_1 : h_{1\mathbf{R}} \to h'_{1\mathbf{R}}$ such that f_1^t maps Δ'_1 onto Δ_1 can be extended to a Lie-isomorphism $\tilde{f}_1 : \tilde{q} \to \tilde{q}'$. Such an isomorphism is defined by

$$\tilde{f}_1 H_\alpha = H_{\alpha'}$$
$$\tilde{f}_1 E_\alpha = E_{\alpha'},$$

where $\alpha = f_1^t \alpha'$ and the $E'_\alpha s$ satisfy $B(E_\alpha, E_{-\alpha}) = 1$.
The same construction may be done with a map $f_0 : \mathrm{h}_{0\mathbf{R}} \to \mathrm{h}'_{0\mathbf{R}}$ (with the same hypothesis), whose extension $\tilde{f}_0$ maps q_0 onto q'_0.

Theorem 4.5.1 *Let* (g, q) *and* $(\mathrm{g}', \mathrm{q}')$ *be CR-semisimple LCR-algebras,* h *and* h' *their Cartan sub-LCR-algebra. Let* Δ *and* Δ' *denote the corresponding CR-root systems. Suppose* $f : \mathrm{h}_\mathbf{R} \to \mathrm{h}'_\mathbf{R}$ *be a* $\mathbf{R}$*-linear one-to-one map such that* $f\mathrm{h}_{j\mathbf{R}} \subseteq \mathrm{h}'_{j\mathbf{R}}$ *and* f^t *maps* Δ'_j *onto* Δ_j*. Then* f *can be extended to a Lie-isomorphism* $\tilde{f} : \mathrm{g} \to \mathrm{g}'$*, which sends* $\tilde{\mathrm{q}}$ *in* $\tilde{\mathrm{q}}'$ *and* q_0 *in* q'_0*.*

Proof: consider the restrictions $f_j = f|_{\mathrm{h}_{j\mathbf{R}}}$, $j = 0, 1$. Both of them admits an extension $\tilde{f}_j$. Define $\tilde{f} \doteq \tilde{f}_1 \oplus \tilde{f}_0$. A direct computation shows that $\tilde{f}$ is a Lie-homomorphism. ∎

Theorem 4.5.2 *For each nonvanishing CR-root* α*, there is a vector* X_α *such that*

$$[H, X_\alpha] = \alpha(H) X_\alpha$$

$$[X_\alpha, X_\beta] = \begin{cases} H_\alpha & \text{if } \beta = -\alpha \\ 0 & \text{if } \alpha + \beta \notin \Delta \\ N_{\alpha,\beta} X_{\alpha+\beta} & \text{if } \alpha + \beta \in \Delta. \end{cases}$$

where $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$*.* ∎

Consider now a generic complex Lie-algebra $\mathfrak{g}$. It may be thought as a real Lie-algebra $\mathfrak{g}^{\mathbf{R}}$ endowed with a complex structure $J_{\mathbf{R}}$ given by the multiplication by i.

Definition 4.5.3 *A real form $\mathfrak{g}_0$ of $\mathfrak{g}$ is a real subalgebra of $\mathfrak{g}^{\mathbf{R}}$ such that $\mathfrak{g}^{\mathbf{R}} = \mathfrak{g}_0 \oplus J_{\mathbf{R}}\mathfrak{g}_0$. A real CR-form of the LCR-algebra $\mathfrak{g}$ is a pair $(\mathfrak{g}_0, \mathfrak{p}_0)$ such that $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$ and $\mathfrak{p}_0$ is a real form of $\grave{\mathfrak{q}}$. A real CR-form $(\mathfrak{g}_0, \mathfrak{p}_0)$ is said to be CR-compact if $\mathfrak{p}_0$ is a compact subalgebra.*

Theorem 4.5.4 *Every CR-semisimple LCR-algebra admits a CR-compact real CR-form.*

Proof: the real subspaces

$$\mathfrak{g}_0^* = \sum_{\alpha\in\Delta} \mathbf{R}iH_\alpha \oplus \sum_{\alpha\in\Delta} \mathbf{R}(X_\alpha - X_{-\alpha}) \oplus \sum_{\alpha\in\Delta} i\mathbf{R}(X_\alpha + X_{-\alpha})$$

$$\mathfrak{p}^* = \sum_{\alpha\in\Delta_1} \mathbf{R}iH_\alpha \oplus \sum_{\alpha\in\Delta_1} \mathbf{R}(X_\alpha - X_{-\alpha}) \oplus \sum_{\alpha\in\Delta_1} i\mathbf{R}(X_\alpha + X_{-\alpha})$$

are Lie-subalgebras, since $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$. By construction, the pair $(\mathfrak{g}_0^*, \mathfrak{p}^*)$ is a real CR-form. Finally, we may compute, with respect of (H_α, X_α), that $B|_{\mathfrak{p}^*\times\mathfrak{p}^*}$ is negative definite. So, $\mathfrak{p}^*$ is compact. ∎

Thus, in the real terms, the classification of the LCR-structures $(\mathfrak{g}_0, \mathfrak{p}, \mathbf{J})$ given on a semisimple ideal is equivalent to the classification of the real Lie-algebras $\mathfrak{g}_0^*$ which admit an even-dimensional compact semisimple ideal $\mathfrak{p}^*$. In fact, if $\mathfrak{p}^*$ is a compact semisimple ideal of $\mathfrak{g}_0$, $\tilde{\mathfrak{q}} \doteq \mathfrak{p}^* \otimes_{\mathbf{R}} \mathbf{C}$ is a semisimple ideal of $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbf{R}} \mathbf{C}$ which admits $\mathfrak{p}^*$ as compact real form. So, if J denotes the multiplication by i, $\tilde{\mathfrak{q}}$ is equal to $\mathfrak{p}^* \oplus J\mathfrak{p}^*$. Hence, the subspace $\mathfrak{q}$ of the elements $x - iJx$ is a complex ideal of $\mathfrak{g}$ which does not intersect $\overline{\mathfrak{q}}$. Then, the set of CR-semisimple LCR-algebras and the one of real Lie-algebras with an even-dimensional semisimple compact ideal, are bijective.

4.6 Appendix.

1. Let us remind that $\mathfrak{g}$ is CR-simple if any nontrivial LCR-ideal contains $\tilde{\mathfrak{q}}$. In particular, $\mathfrak{q}$ is simple. The vice versa is also true. In fact, whenever $\mathfrak{q}$ is simple, any LCR-ideal $\mathfrak{h}$ of $\mathfrak{g}$ contains $\tilde{\mathfrak{q}}$. thus, $\mathfrak{g}$ is CR-simple. Obviously, a CR-simple LCR-algebra is CR-semisimple.

Theorem. *Let $\mathfrak{g}$ be a CR-semisimple LCR-algebra and $\mathfrak{q}$ be decomposed as $\mathfrak{q} = \mathfrak{q}_1 \odot \ldots \odot \mathfrak{q}_k$. then, there exist some LCR-ideal $\mathfrak{g}_1$ such that*

1. $\mathfrak{g} = \mathfrak{g}_1 \odot \ldots \odot \mathfrak{g}_k$;
2. $\mathfrak{g}_i \cap \mathfrak{q} = \mathfrak{q}_i$;
3. $\mathfrak{g}_i$ *is CR-simple.* ■

Furthermore, we link the CR-simplicity and the CR-maximality, via the following

Proposition. *A CR-simple LCR-algebra is CR-maximal.*

The proof is a direct consequence of Theorem 3.7.4.

Thus, a CR-simple LCR-algebra $\mathfrak{g}$ satisfies the following properties:

1. $\mathfrak{g}$ is reductive;
2. its center $\zeta(\mathfrak{g})$ has dimension less then two;
3. its semisimple part $\mathcal{D}\mathfrak{g}$ is the sum of two, three or four simple ideals.

In particular,

$$
\mathrm{g} = \begin{cases} \mathrm{q} \odot \overline{\mathrm{q}} \\ \mathrm{q} \odot \overline{\mathrm{q}} \odot \mathrm{C}H \\ \mathrm{q} \odot \overline{\mathrm{q}} \odot \mathrm{C}H \odot \mathrm{C}\overline{H} \\ \mathrm{q} \odot \overline{\mathrm{q}} \odot \mathrm{h} \\ \mathrm{q} \odot \overline{\mathrm{q}} \odot \mathrm{h} \odot \overline{\mathrm{h}} \end{cases}
$$

2. Take, now, a CR-semisimple LCR-algebra g endowed with its CR-root set Δ.

Lemma. *the set Δ is a reduced root system of the Cartan sub-LCR-algebra* h.

Proof: by definition, Δ spans h^*. Moreover, consider the reflection $S_\alpha\beta \doteq \beta - 2\frac{<\alpha,\beta>}{<\alpha,\alpha>}\alpha$, where $<\alpha,\beta>=\alpha(H_\beta)$. By Proposition 4.3.4, S_α maps Δ onto Δ and the number $a_{\alpha\beta} = -2\frac{<\alpha,\beta>}{<\alpha,\alpha>}$ is a integer. Finally, if $m\alpha$ is a root, $m=-1$. ■

The root system Δ is no irreducible, in fact

$\Delta = \Delta_0 \cup \Delta_1$

$< \Delta_0, \Delta_1 >= 0.$

Moreover, $\Delta_1 = \Delta_1(\mathrm{q}) \cup \Delta_1(\overline{\mathrm{q}})$ and $< \Delta_1(\mathrm{q}), \Delta_1(\overline{\mathrm{q}}) >= 0$.

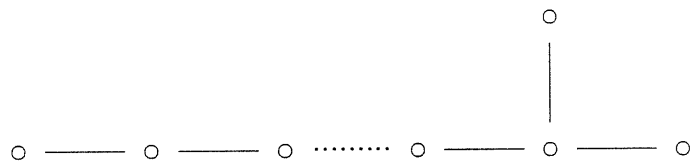
Then, we may consider a simple root system $\Phi = \{\alpha_1, \ldots, \alpha_k\}$ endowed with its Cartan matrix $a_{ij} = -2\frac{<\alpha_i,\alpha_j>}{<\alpha_i,\alpha>_i}$. Via the Cartan matrix, we construct the diagram of g. It consists of a vertex for each α_i, with $a_{ij}a_{ji}$ lines betweew α_i and α_j, $i \neq j$.

Remind that a diagram is connected when Φ is irreducible; and Φ is irreducible if and only if g is simple. The connected diagrams are

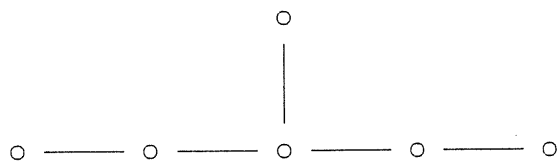
A_l

B_l

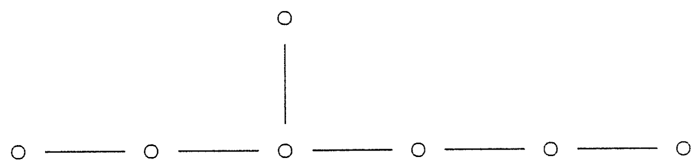
D_l



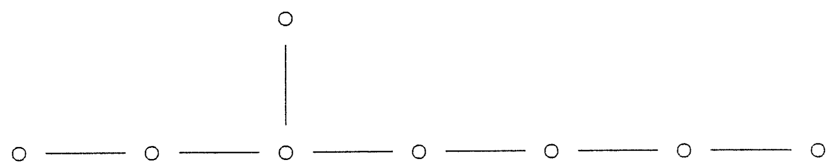
E_6



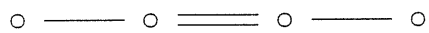
E_7



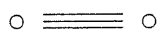
E_8



F_4



G_2



3. In this point, we describe the disconnected diagram of a CR-semisimple LCR-algebra. Let us stars with the Cr-simple case.

A CR-simple LCR-algebra $\mathfrak{g}$ is either semisimple (if it is of the I of the III type) or reductive with center of dimension one or two. This means that the diagram has two connected components (if $\mathfrak{g}$ is of type I or II); while the connected components are three or four, for the type III.

type	$\mathcal{D}\mathfrak{g}$	$\zeta(\mathfrak{g})$	number of components
I	$\mathfrak{g}$	$\{0\}$	2
II	$\mathfrak{q} \odot \overline{\mathfrak{q}}$	$\mathbb{C}H$	2
		$\mathbb{C}H \odot \mathbb{C}\overline{H}$	
III	$\mathfrak{g}$	$\{0\}$	3
			4

Finally, the disconnected diagram of a CR-semisimple LCR-algebra is the disjoint union of the diagram of its CR-simple LCR-ideals.

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