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Lie-CR-structures on a real Lie algebra

Daniele Gouthier [1]
S.I.S.S.A.–I.S.A.S., via Beirut 2, 4–34014 Trieste, Italy

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Abstract: Let $\mathbf{g}_0$ be a real Lie algebra and $\mathbf{g}$ its complexification. The aim of this paper is to study the Lie-CR-structures (in the following we shall call them just LCR-structures) on $\mathbf{g}_0$. To an LCR-structure corresponds a CR-structure on the associated real Lie group $\mathbf{G}_0$ for which right and left translations are both CR-maps. Levi–Mal'cev decomposition permits to consider separately the semisimple and the solvable cases and to describe completely the LCR-structures of a generic real Lie algebra $\mathbf{g}_0$. Hence, we introduce and describe the semidirect sum by the adjoint derivation of the structures induced on the solvable radical and on a Levi subalgebra.

Keywords: Lie algebras, Lie groups, CR-structures.

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Introduction

The aim of this paper is to classify the *Lie-CR-structures* of a real Lie algebra $\mathbf{g}_0$. By an LCR-structure on a real Lie algebra $\mathbf{g}_0$ we mean a triple $\Gamma_0 = (\mathbf{g}_0, \mathbf{p}, J)$ such that $\mathbf{p}$ is an ideal in $\mathbf{g}_0$ and J is an endomorphism of $\mathbf{p}$ whose square is minus the identity (so $\mathbf{p}$ has to be even-dimensional) and which commutes with the adjoint derivations ad_X, $\forall X \in \mathbf{g}_0$.

Of course LCR-structures are a particular kind of CR-structures [4]. Recall that a *CR-structure* $(\mathbf{p}, J)$ on a real Lie algebra $\mathbf{g}_0$ is given by a linear subspace $\mathbf{p}$ and by an endomorphism $J : \mathbf{p} \to \mathbf{p}$ such that

1. $J^2 = -\,\mathrm{id}$,
2. $[X, Y] - [JX, JY] \in \mathbf{p}, \forall X, Y \in \mathbf{p}$,
3. $[JX, JY] = [X, Y] + J[X, JY] + J[JX, Y], \forall X, Y \in \mathbf{p}$.

If one denotes with $\mathbf{g}$ the complexification of $\mathbf{g}_0$ and with τ the conjugation of $\mathbf{g}$ with respect to $\mathbf{g}_0$, then the linear space $\mathbf{q} = \{X - iJX : X \in \mathbf{p}\}$ is really a complex subalgebra of $\mathbf{g}$ which does not intersect its conjugate $\bar{\mathbf{q}} = \tau\mathbf{q}$. Given such a $\mathbf{q}$, $\mathbf{g}$ can be written as a direct sum $\mathbf{g} = \mathbf{q} \oplus \bar{\mathbf{q}} \oplus V$, where $V = \bigoplus_{i=1}^{r} \mathbb{C}X_i$, $X_1 \dots X_r \in \mathbf{g}_0$. The integer r is called the *real codimension* of the CR-structure.

It is quite easy to introduce basic CR-concepts: CR-subalgebra, CR-ideal, CR-homomorphism, equivalence between CR-structures. In fact, consider a subalgebra $\mathbf{h}_0$ in $\mathbf{g}_0$, a natural question is "when does $\mathbf{h}_0$ assume a CR-structure induced by Γ_0?" Let $\mathbf{h}$ be its complexification. Let us denote by $\mathbf{k}$ and $\tilde{\mathbf{k}}$ the subalgebras $\mathbf{h} \cap \mathbf{q}$ and $\mathbf{h} \cap \bar{\mathbf{q}}$, respectively. Obviously, $\mathbf{k} \cap \tilde{\mathbf{k}} = \mathbf{k} \cap (\mathbf{h} \cap V) = \tilde{\mathbf{k}} \cap (\mathbf{h} \cap V) = \{0\}$. But, in the general case, $\mathbf{k}$ and $\tilde{\mathbf{k}}$ are not isomorphic. Take, for instance, $\mathbf{h} = \mathbf{q}$,

[1] *E-mail*: gouthier@sissa.it.

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then you have $\mathbf{k} = \mathbf{h}$ and $\widetilde{\mathbf{k}} = \{0\}$. So we are forced to say that *a real subalgebra* $\mathbf{h}_0$ *admits the CR-structure* $(\mathbf{h}_0, \mathbf{k}, J)$ *induced by* Γ_0 if $\mathbf{k} \neq \{0\}$ and $\tau\mathbf{k} = \widetilde{\mathbf{k}}$. In that case $\mathbf{h}$ is said a *CR-subalgebra* in $\mathbf{g}$. Let $\mathbf{h}_0$ be an ideal, then $\mathbf{h}$ is said a *CR-ideal*. One can easily notice that $\mathbf{q}$ is not a CR-subalgebra; every subalgebra of V is a trivial CR-subalgebra; let $\mathbf{h}$ be a CR-subalgebra which doesn't intersect V, then $\mathbf{k}$ is an almost complex structure on $\mathbf{h}_0$. In order to compare different CR-structures, we introduce the *CR-homorphisms*: given two CR-structures, $\Gamma_0 = (\mathbf{g}_0, \mathbf{p_g}, J_{\mathbf{g}_0})$ and $\Xi_0 = (\mathbf{h}_0, \mathbf{p_h}, J_{\mathbf{h}_0})$, a Lie homomorphism (resp. a Lie derivation) $\sigma : \mathbf{g}_0 \to \mathbf{h}_0$ is a *CR-homomorphism* (resp. a *CR-derivation*) if $\sigma \mathbf{p_g} \subset \mathbf{p_h}$ and $\sigma J_{\mathbf{g}_0} = J_{\mathbf{h}_0}\sigma$.

Now, let $\Gamma_0 = (\mathbf{g}_0, \mathbf{p_g}, J)$ and $\Xi_0 = (\mathbf{h}_0, \mathbf{p_h}, J')$ be CR-structures and $\varphi : \mathbf{g}_0 \to \mathbf{h}_0$ a CR-monomorphism. In that case, we say that Γ_0 is *φ-compatible* with Ξ_0 and that Γ_0 is a *φ-CR-substructure* of H_0. Hence, if one considers the triple $\varphi^*\Gamma_0 = (\varphi\mathbf{g}_0, \varphi\mathbf{p_g}, \varphi J\varphi^{-1})$, one obtains a CR-structure. It is a fact that Γ_0 is φ-compatible with Ξ_0 if and only if $\varphi\mathbf{g}$ is a CR-subalgebra of $\mathbf{h}$. We can say that Γ_0 and Ξ_0 are *equivalent* (or *CR-isomorphic*) if there exist $\varphi : \mathbf{g}_0 \to \mathbf{h}_0$ and $\psi : \mathbf{h}_0 \to \mathbf{g}_0$ such that Γ_0 is φ-compatible with Ξ_0 and Ξ_0 is ψ-compatible with Γ_0. In that case, $\mathbf{g}_0$ and $\mathbf{h}_0$ are Lie isomorphic (via φ and ψ); φ (resp. ψ) sends $\mathbf{p_g}$ in $\mathbf{p_h}$ (resp. $\mathbf{p_h}$ in $\mathbf{p_g}$); and $\varphi J_{\mathbf{g}_0} = J_{\mathbf{h}_0}\varphi$. Obviously we can replace the ψ-compatibility with the φ^{-1}-compatibility. Two equivalent CR-structures are given, for instance, by Γ_0 and $\varphi^*\Gamma_0$, for any CR-monomorphism from Γ_0 in another CR-structure Ξ_0.

If $\mathbf{G}_0$ is the associated Lie group, giving a CR-structure Γ_0 on $\mathbf{g}_0$ is the same as considering a structure of CR-manifold on $\mathbf{G}_0$ for which the left translations are CR-maps.

If the CR-structure Γ_0 is such that $\mathbf{p}$ is a real Lie subalgebra (i.e. $\mathbf{q} \oplus \bar{\mathbf{q}}$ is a complex one), the Lie group $\mathbf{G}_0$ with the CR-structure Γ_0 is a Levi-flat manifold: such CR-manifolds have Levi-form vanishing at each point, [1,2,13]. Moreover, if $\mathrm{ad}_X J = J\,\mathrm{ad}_X$ for all $X \in \mathbf{g}_0$ and $\mathbf{p}$ is an ideal (i.e., $\mathbf{q}$ is it), then $\mathbf{p}$ (with the complex structure J) is, really, a complex subalgebra; and the Lie group $\mathbf{G}_0$ is foliated with complex Lie subgroups and both the translations are CR-maps. Such a case is studied in this paper. In these conditions we say that the CR-structure Γ_0 is a *LCR-structure*.

In [11] the reader can find a study on left-invariant complex structure on reductive Lie groups. Such results have been translated by [4] in terms of CR-structures on reductive Lie algebras of the first category (in these algebras the involution determined by a Cartan decomposition is an inner automorphism [6]): the authors study, essentially, the case of real codimension 1.

In this paper we explore the case of LCR-structures without any hypothesis on the Lie algebra. This exploration permits us to classify all the LCR-structures. We base our work on the classical Levi–Mal'cev theorem which assures that all the Lie algebras admit a decomposition $\mathbf{g}_0 = \mathbf{r} \oplus_{\mathrm{ad}} \mathbf{s}$, where $\mathbf{r}$ is the solvable radical and $\mathbf{s}$ is a semisimple subalgebra [12]. Remark that we denote with $\oplus$ the direct sum of linear spaces; with $\oplus_\delta$ the semidirect sum by δ of Lie algebras; with $\odot$ the direct sum of Lie algebras.

Thanks to Levi–Mal'cev decomposition, we have to study LCR-structures in the semisimple and in the solvable cases (Sections 1 and 2): in the first one the LCR-structures are sums (in the sense of the last proposition of Section 1) of simple ideals endowed with a complex structure (described by Cartan in the classical classification [6]); in the second one they are given by even-dimensional ideals $\mathbf{p}$, decomposed as $\mathbf{p} = \mathbf{u} \oplus A\mathbf{u}$ and by

$$J = J_A := \begin{pmatrix} 0 & -A^{-1} \\ A & 0 \end{pmatrix},$$

endomorphism of $\mathbf{p}$. Section 3 shall proof when there is an LCR-structure on a semidirect sum whose factors are endowed with two LCR-structures. Finally, Section 4 concludes with the Theorem 4: let $\mathbf{g}_0$ be decomposed following Levi–Mal'cev decomposition. Then $\Gamma_0 = (\mathbf{g}_0, \mathbf{p}, J)$ is an LCR-structure if and only if its projections on the factors are LCR-structures whose semidirect sum by the adjoint derivation ad is Γ_0 itself. Obviously this result let us describe all the LCR-structures. The only indetermination is due to the knowledge of the ideals of solvable Lie algebras. The last section is devoted to some examples.

1. The semisimple case

In this Section we denote by $\mathbf{g}_0$ a real Lie algebra and by B its Killing form (recall that if $\mathbf{g}_0$ is semisimple, B is nondegenerate). Just by computation, one can observe that J is an anti-isometry with respect to B. In fact, $\mathrm{ad}_{JX} Y = -\mathrm{ad}_Y JX = -J\,\mathrm{ad}_Y X = J\,\mathrm{ad}_X Y$. So we have $\mathrm{ad}_{JX} = \mathrm{ad}_X J = J\,\mathrm{ad}_X$ and $B(JX, JY) = -B(X, Y)$.

In the sequel we shall divide our study in two subcases: the first one considers compact semisimple real Lie algebras; the second one those noncompact. Recall that a Lie algebra $\mathbf{g}_0$ is *compact* if there exists a compact Lie group whose Lie algebra is $\mathbf{g}_0$. That is equivalent to the decomposition $\mathbf{g}_0 = \zeta(\mathbf{g}_0) \odot [\mathbf{g}_0, \mathbf{g}_0]$, where $\zeta(\mathbf{g}_0)$ is the center of $\mathbf{g}_0$ and $[\mathbf{g}_0, \mathbf{g}_0]$ is semisimple [5].

It is a classical fact that the existence of a complex structure on a compact (not semisimple) Lie algebra implies the abelianity of the algebra itself [5]. Moreover, if $\mathbf{p}$ is in the center of $\mathbf{g}_0$, Γ_0 is trivially an LCR-structure, so we can hopefully expect a CR analogous of the complex result. Such an analogous result is based on the

Lemma. *Given $\Gamma_0 = (\mathbf{g}_0, \mathbf{p}, J)$ LCR-structure, $\mathbf{p}$ admits a biinvariant metric if and only if $\mathbf{p}$ is abelian.*

Proof. A metric g is biinvariant iff $g([X, Y], Z) = g(X, [Y, Z])$. Suppose $\mathbf{p}$ is abelian, then any metric is, certainly, biinvariant. Let us prove the converse. We can impose that J is an isometry with respect to g (otherwise we substitute g with $g'(X, Y) := g(X, Y) + g(JX, JY)$). With this hypothesis the following chain of equivalences is true, $\forall X, Y, Z \in \mathbf{p}$: $g([X, Y], Z) = g(J[X, Y], JZ) = g([X, JY], JZ) = g(X, [JY, JZ]) = -g(X, [Y, Z]) = -g([X, Y], Z) = 0$, therefore $g([X, Y], Z) = 0$, which concludes the lemma. □

Since any compact Lie algebra admits a biinvariant metric the above lemma implies the following

Proposition. *Given $\mathbf{g}_0$ compact (not semisimple), $\Gamma_0 = (\mathbf{g}_0, \mathbf{p}, J)$ is an LCR-structure if and only if $\mathbf{p}$ is abelian.*

The previous proposition permits us to describe completely the compact case with the

Theorem 1. *There are no LCR-structures on a compact semisimple Lie algebra. Furthermore, when $\mathbf{g}_0$ is compact, $\Gamma_0 = (\mathbf{g}_0, \mathbf{p}, J)$ is an LCR-structure if and only if $\mathbf{p}$ is in the center $\zeta(\mathbf{g}_0)$.*

Proof. The nonexistence of abelian ideals in a semisimple Lie algebra concludes the first part of the assertion. About the second one, recall that a compact Lie algebra $\mathbf{g}_0$ takes the form

$\mathbf{g}_0 = \zeta(\mathbf{g}_0) \odot [\mathbf{g}_0, \mathbf{g}_0]$, where $[\mathbf{g}_0, \mathbf{g}_0]$ is semisimple. Now suppose that $\mathbf{g}_0$ supports an LCR-structure $\Gamma_0 = (\mathbf{g}_0, \mathbf{p}, J)$, then $\mathbf{p}$ takes the form $\mathbf{p}_1 \oplus \mathbf{p}_2$ where $\mathbf{p}_2$ is an ideal in $[\mathbf{g}_0, \mathbf{g}_0]$. Suppose that J maps $\mathbf{p}_2$ in itself, then $([\mathbf{g}_0, \mathbf{g}_0], \mathbf{p}_2, J)$ is a LCR-structure, that is impossible. Hence, $\mathbf{p} = \mathbf{p}_1 \subset \zeta(\mathbf{g}_0)$. Let us conclude proving that J maps $\mathbf{p}_2$ in itself. Consider the complex subalgebras $\mathbf{q}_j = \{X - iJX : X \in \mathbf{p}_j\}$. Obviously it is $\mathbf{q} = \mathbf{q}_1 \oplus \mathbf{q}_2$ and $\mathbf{q}_2$ is another LCR-structure of $\mathbf{g}_0$. Hence, it is given the endomorphism $J_2 : \mathbf{p}_2 \to \mathbf{p}_2$. Take $X \in \mathbf{p}_2$, then $X - iJX \in \mathbf{q}$ (resp. $X + iJX \in \bar{\mathbf{q}}$) and $X - iJ_2X \in \mathbf{q}_2 \subset \mathbf{q}$ (resp. $X + iJ_2X \in \bar{\mathbf{q}}_2 \subset \bar{\mathbf{q}}$), so $i(J_2X - JX) = (X - iJX) - (X - iJ_2X) = (X + iJ_2X) - (X + iJX) \in \mathbf{q} \cap \bar{\mathbf{q}} = \{0\}$, which means that J maps $\mathbf{p}_2$ in itself. $\square$

In Section 2 we shall study LCR-structures on abelian Lie algebras. Hence we will describe completely LCR-structures in the compact case. Now we move to the study of LCR-structures on semisimple noncompact Lie algebras. The simple case is trivial. In fact, since there are no nontrivial ideals, an LCR-structure on a simple Lie algebra is, really, an ad-invariant complex one. Moreover, it is well known that a semisimple Lie algebra is direct sum of simple ideals. These facts bring us to the

Proposition. *An LCR-structure on a semisimple Lie algebra is completely defined by its simple ideals endowed with a complex structure.*

Proof. Since $\mathbf{g}_0$ is semisimple, we can write $\mathbf{g}_0 = \mathbf{p}_1 \odot \cdots \odot \mathbf{p}_k$, where the $\mathbf{p}_j$ are simple ideals (such a decomposition is essentially unique). Let $\mathbf{p}$ be the ideal on which is given the CR-structure, then $\mathbf{p}$ is direct sum of some $\mathbf{p}_i$. As in Theorem 1, the restriction of J to $\mathbf{p}_i$ has image in $\mathbf{p}_i$. This fact concludes the proof. $\square$

Hence, an LCR-structure on a semisimple Lie algebra is given by the complex structures on some simple factors. Each of these factors is described in the Cartan's classification of the complex simple Lie algebras

$\mathbf{g}$	$\mathbf{G}$	$\mathbf{U}$	$\zeta(\mathbf{U}')$	$\dim \mathbf{U}$
a_n $(n \geqslant 1)$	$\mathbf{SL}(n+1, \mathbb{C})$	$\mathbf{SU}(n+1)$	$\mathbb{Z}_{n+1}$	$n(n+2)$
b_n $(n \geqslant 2)$	$\mathbf{SO}(2n+1, \mathbb{C})$	$\mathbf{SO}(2n+1)$	$\mathbb{Z}_2$	$n(2n+1)$
c_n $(n \geqslant 3)$	$\mathbf{Sp}(n, \mathbb{C})$	$\mathbf{Sp}(n)$	$\mathbb{Z}_2$	$n(2n+1)$
d_n $(n \geqslant 4)$	$\mathbf{SO}(2n, \mathbb{C})$	$\mathbf{SO}(2n)$	$\mathbb{Z}_4$, $n =$ odd $\mathbb{Z}_2 + \mathbb{Z}_2$, $n =$ even	$n(2n-1)$
e_6	$E_6^{\mathbb{C}}$	E_6	$\mathbb{Z}_3$	78
e_7	$E_7^{\mathbb{C}}$	E_7	$\mathbb{Z}_2$	133
e_8	$E_8^{\mathbb{C}}$	E_8	$\mathbb{Z}_1$	248
f_4	$F_4^{\mathbb{C}}$	F_4	$\mathbb{Z}_1$	52
g_2	$G_2^{\mathbb{C}}$	G_2	$\mathbb{Z}_1$	14

Table 1.

In the table, $\mathbf{g}$ is a simple Lie algebra over $\mathbb{C}$; n the dimension of a Cartan-subalgebra; $\mathbf{G}$ a connected Lie group such that $\text{Lie}(\mathbf{G}) = \mathbf{g}^{\mathbb{R}}$; $\mathbf{U}$ an analytical subgroup such that $\text{Lie}(\mathbf{U})$ is a compact real form of $\mathbf{g}$ (i.e., $\mathbf{U}$ is a maximal compact subgroup); and $\mathbf{U}'$ is its universal covering [6].

The present section will be concluded by the complete description of LCR-structures on semisimple Lie algebras. As already showed in Theorem 1, when the algebra is compact there are no LCR-structures. Otherwise, is given the following

Classification. *Let* $\mathbf{g}_0$ *be semisimple and noncompact. Then we give the (essentially unique) decomposition* $\mathbf{g}_0 = \mathbf{r}_1 \odot \cdots \odot \mathbf{r}_j \odot \mathbf{p}_1 \odot \cdots \odot \mathbf{p}_h$, *where*:

1. *both* $\mathbf{r}_i$ *and* $\mathbf{p}_i$ *are simple real ideals*;
2. *on the* $\mathbf{r}_i$ *there are no complex structures*;
3. *any* $\mathbf{p}_i$ *takes one of the forms in Table* 1.

With such a decomposition we may choose any sum $\mathbf{p} = \bigodot_{l=1}^{k} \mathbf{p}_{i_l}$ *with the endomorphism*

$$J = \begin{pmatrix} J_{i_1} & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & J_{i_k} \end{pmatrix}.$$

The triple $(\mathbf{g}_0, \mathbf{p}, J)$ *is the generic LCR-structure in the semisimple case.*

2. The solvable case

A real Lie algebra $\mathbf{g}_0$ is *solvable* if one of its *derived* subalgebras vanishes. Recall that the *derived series* is given by $\mathbf{D}^0\mathbf{g}_0 = \mathbf{g}_0$, $\mathbf{D}^1\mathbf{g}_0 = [\mathbf{g}_0, \mathbf{g}_0]$, $\mathbf{D}^n\mathbf{g}_0 = \mathbf{D}^1(\mathbf{D}^{n-1}\mathbf{g}_0)$. Since any ideal of $\mathbf{g}_0$ is solvable, an LCR-structure on $\mathbf{g}_0$ is an ad-invariant complex structure on a solvable ideal. Let us study such structures on solvable Lie algebras. In the sequel, when we speak of a complex structure, we mean an ad-*invariant* one.

Lemma. *Suppose* $\mathbf{g}_0$ *is solvable and* $(\mathbf{g}_0, J)$ *is a complex structure. Then there exists a subspace* $\mathbf{u}$ *such that* $\mathbf{g}_0 = \mathbf{u} \oplus J\mathbf{u}$ *and*

$$J = \begin{pmatrix} 0 & J'' \\ J' & 0 \end{pmatrix},$$

where J' (*resp.* J'') *is the restriction of* J *to* $\mathbf{u}$ (*resp.* $J\mathbf{u}$).

Proof. Since $\mathbf{g}_0$ is solvable there exists a codimension one ideal $\mathbf{p}_1$ [12]. It is easy to show that $J\mathbf{p}_1 \neq \mathbf{p}_1$. Then, there exists $X_1 \in \mathbf{p}_1$ such that $\mathbf{g}_0 = L(X_1, JX_1) \oplus \mathbf{p}_1 \cap J\mathbf{p}_1$. Moreover $(\mathbf{g}_0, \mathbf{p}_1 \cap J\mathbf{p}_1, J)$ is an LCR-structure. Now we repeat the same proof with $\mathbf{p}_1 \cap J\mathbf{p}_1$ and $\mathbf{p}_2$ (where $\mathbf{p}_2$ is a codimension one ideal in $\mathbf{p}_1 \cap J\mathbf{p}_1$) instead of $\mathbf{g}_0$ and $\mathbf{p}_1$. In that way, we find a family $X_1, \ldots, X_k$, such that $\mathbf{g}_0 = L(X_1, \ldots, X_k, JX_1, \ldots, JX_k)$ and the space $\mathbf{u} = L(X_1, \ldots, X_k)$ is the desired one. $\square$

Now we want to show the converse, in the sense that any solvable Lie algebra admits a complex structure if and only if it is even-dimensional; in that case we write $\mathbf{g}_0$ as the sum $\mathbf{g}_0 = \mathbf{u} \oplus \mathbf{v}$, where $\mathbf{u}$ and $\mathbf{v}$ have the same dimension. Chosen a linear monomorphism $A : \mathbf{u} \to \mathbf{g}_0$ such that $\mathbf{v} = A\mathbf{u}$, the complex structure

$$J = J_A := \begin{pmatrix} 0 & -A^{-1} \\ A & 0 \end{pmatrix}$$

is generic: so the complex structures depend only on the splitting of $\mathbf{g}_0$ in equal-dimensional subspaces. Let us prove this fact by induction.

The simplest solvable algebras are the abelian ones, i.e., the ones whose first derived vanishes. On these, a complex structure is just the "multiplication by i," in fact you have the

Lemma. *Let $\mathbf{g}_0$ be abelian. Then there exists a complex structure J if and only if $\mathbf{g}_0$ is even-dimensional. In that case there exist a linear subspace $\mathbf{u}$ and a linear monomorphism $A : \mathbf{u} \to \mathbf{g}_0$ such that*

1. $\mathbf{g}_0 = \mathbf{u} \oplus A\mathbf{u}$,
2. $J = J_A := \begin{pmatrix} 0 & -A^{-1} \\ A & 0 \end{pmatrix}$.

Moreover, all the pair $(\mathbf{g}_0, J_A)$ are isomorphic as complex Lie algebras, independently on the subspace $\mathbf{u}$ and the morphism A. So we may say that the structure is unique.

Proof. Suppose that $\mathbf{g}_0$ is endowed with a complex structure J, then previous lemma gives us the pair $(\mathbf{u}, J')$ desired. Vice versa, let $\mathbf{g}_0$ be even-dimensional. Then, choose $\mathbf{u}$ and A, such that $\mathbf{g}_0 = \mathbf{u} \oplus A\mathbf{u}$. The endomorphism J_A is trivially a complex structure on $\mathbf{g}_0$. If one considers the automorphism

$$\phi_{AB} := \begin{pmatrix} I & 0 \\ 0 & BA^{-1} \end{pmatrix},$$

one has an isomorphism between $(\mathbf{g}_0, J_A)$ and $(\mathbf{g}_0, J_B)$. Hence, the complex structure does not depend on A. Finally, we show that does not depend neither on $\mathbf{u}$: let $(\mathbf{v}, C)$ be a pair such that $\mathbf{g}_0 = \mathbf{v} \oplus C\mathbf{v}$. Then we have $\mathbf{v} = D\mathbf{u}$ and $\mathbf{g}_0 = D\mathbf{u} \oplus AD\mathbf{u}$, where we have taken D Lie isomorphism. It is easy to show that $(D\mathbf{u} \oplus AD\mathbf{u}, J_A)$ and $(\mathbf{u} \oplus D^{-1}AD\mathbf{u}, J_{D^{-1}AD})$ are isomorphic. □

In the previous section we have shown that, given a compact Lie algebra $\mathbf{g}_0$, $\Gamma_0 = (\mathbf{g}_0, \mathbf{p}, J)$ is an LCR-structure if and only if $\mathbf{p}$ is contained in the center $\zeta(\mathbf{g}_0)$. The last property permits us to describe these LCR-structures. In fact, suppose $\Gamma_0 = (\mathbf{g}_0, \mathbf{p}, J)$ is an LCR-structure, then $\mathbf{p}$ has to be even-dimensional and takes the form $\mathbf{p} = \mathbf{u} \oplus A\mathbf{u}$, with $J = J_A$. Finally, the datum of an LCR-structure on a compact Lie algebra is equivalent to the choice of an even-dimensional linear subspace of the center. Let us return to solvable algebras.

Theorem 2. *A solvable Lie algebra $\mathbf{g}_0$ admits a complex structure if and only if it is even-dimensional. Let $(\mathbf{g}_0, J)$ be a complex structure, then there exist two vector spaces $\mathbf{u}$ and $\mathbf{v}$ and an isomorphism A between $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{g}_0 = \mathbf{u} \oplus A\mathbf{u}$ and $J = J_A$. Moreover, all the pairs $(\mathbf{g}_0, J_A)$ are isomorphic as complex Lie algebras.*

Proof. Let k be the minimum integer such that $\mathbf{D}^k\mathbf{g}_0 = 0$, then make the proof by induction over k. The base of the induction is given by the abelian case. Now, let $\mathbf{g}_0$ be solvable but not abelian. In any case, $\hat{\mathbf{g}}_0 := \mathbf{g}_0/\mathbf{D}^1\mathbf{g}_0$ is abelian. Furthermore J maps $\mathbf{D}^1\mathbf{g}_0$ on itself, since $J\,\mathrm{ad}_X = \mathrm{ad}_X\,J$. So $(\hat{\mathbf{g}}_0, \hat{J})$ is a complex structure, where $\hat{J}$ is the quotient of J. If we apply the previous lemma, we have that $\hat{\mathbf{g}}_0 = \hat{\mathbf{w}} \oplus \hat{J}|_{\hat{\mathbf{w}}}\hat{\mathbf{w}}$ and $\hat{J} = J_{\hat{J}|_{\hat{\mathbf{w}}}}$. If we choose a subspace $\mathbf{w}$ in the class $\hat{\mathbf{w}}$, we obtain $\mathbf{g}_0 = \mathbf{w} \oplus J^+\mathbf{w} \oplus \mathbf{D}^1\mathbf{g}_0$ and

$$J = \begin{pmatrix} 0 & J^+ & 0 \\ J^+ & 0 & 0 \\ 0 & 0 & J_1 \end{pmatrix},$$

where J^+ is the restriction at $\mathbf{w}$ and J_1 the one at $\mathbf{D}^1\mathbf{g}_0$. Finally, we apply the inductive hypothesis on the pair $(\mathbf{D}^1\mathbf{g}_0, J_1)$. □

Let $\mathbf{g}_0$ be solvable, then $\mathbf{g}_0$ admits one LCR-structure on each $2l$-dimensional ideal (in the hypothesis that it exists) given by an isomorphism J of the form J_A. Hence LCR-structures are essentially given by the choice of even-dimensional ideals. Remark that it is possible to have different LCR-structures of the same dimension. The study of LCR-structures on a solvable real Lie algebra is now completely equivalent to the knowledge of the ideals of the algebra itself: all even-dimensional ideals support one, and only one, LCR-structure.

3. Semidirect sums

Take two LCR-structures $\Gamma_0 = (\mathbf{g}_0, \mathbf{p}, J)$ and $\Gamma_0' = (\mathbf{g}_0', \mathbf{p}', J')$. Let $\delta : \mathbf{g}_0' \to \mathrm{Der}(\mathbf{g}_0)$ be a Lie homomorphism. The *semidirect sum* of $\mathbf{g}_0$ and $\mathbf{g}_0'$ by δ is the Lie algebra $\mathbf{g}_0 \oplus_\delta \mathbf{g}_0'$ defined on the linear space $\mathbf{g}_0 \oplus \mathbf{g}_0'$ by the product $[(X, X'), (Y, Y')]_\delta = ([X, Y] + \delta(X')Y - \delta(Y')X, [X', Y'])$. The actual problem is whether the triple

$$\Gamma_0 \oplus_\delta \Gamma_0' = \left(\mathbf{g}_0 \oplus_\delta \mathbf{g}_0',\ \ \mathbf{p}^* = \mathbf{p} \oplus \mathbf{p}',\ \ J^* = \begin{pmatrix} J & 0 \\ 0 & J' \end{pmatrix}\right)$$

is an LCR-structure. When it is, one shall say $\Gamma_0 \oplus_\delta \Gamma_0'$ the *semidirect sum* by δ of Γ_0 and Γ_0'. We will answer by steps.

Lemma. *Given the linear subspaces* $\mathbf{p} \subset \mathbf{g}_0$ *and* $\mathbf{p}' \subset \mathbf{g}_0'$, *their sum* $\mathbf{p}^* = \mathbf{p} \oplus \mathbf{p}'$ *is an ideal if and only if*

(*a*) $\mathbf{p}$ *and* $\mathbf{p}'$ *are ideals*;
(*b*) $\delta(X')\mathbf{g}_0 \subset \mathbf{p}, \forall X' \in \mathbf{p}'$;
(*c*) $\delta(Y')\mathbf{p} \subset \mathbf{p}, \forall Y' \in \mathbf{g}_0'$.

Proof. Let us suppose that (X, X') stays in $\mathbf{p}^*$ and that (Y, Y') is the generic element of $\mathbf{g}_0 \oplus_\delta \mathbf{g}_0'$. The relation $[(X, X'), (Y, Y')]_\delta \in \mathbf{p}^*$ is equivalent to

1. $[X, Y] + \delta(X')Y - \delta(Y')X \in \mathbf{p}$;
2. $[X', Y'] \in \mathbf{p}'$.

Particular choices of (X, X') and (Y, Y') imply that the first one is equivalent to the following three

1.1. $[X, Y] \in \mathbf{p}$;
1.2. $\delta(X')Y \in \mathbf{p}$;
1.3. $\delta(Y')X \in \mathbf{p}$.

The conditions 1.1 and 2. say that $\mathbf{p}$ and $\mathbf{p}'$ are ideals; while 1.2 (resp. 1.3) coincides with letter (b) (resp. (c)). □

Moreover, impose that $\mathrm{ad}_{(X,X')}$ is a CR-derivation, $\forall X \in \mathbf{g}_0$ and $\forall X' \in \mathbf{g}_0'$ (i.e., J^* is ad-invariant). Hence, one obtains the following necessary and sufficient conditions:

(d) ad_X and ad'_X are CR-derivations;
(e) $J\delta(X') = \delta(J'X'), \forall X' \in \mathbf{p}'$;
(f) $\delta(X')$ is a CR-derivation, $\forall X' \in \mathbf{g}_0'$.

The previous computations let us conclude with the

Theorem 3. *Given two LCR-structures Γ_0 and Γ_0', their semidirect sum by δ is an LCR-structure if*

1.1. $\delta(X')\mathbf{g}_0 \subset \mathbf{p}, \forall X' \in \mathbf{p}'$;
1.2. $J\delta(X') = \delta(J'X'), \forall X' \in \mathbf{p}'$;
2.1. $\delta(Y')\mathbf{p} \subset \mathbf{p}, \forall Y' \in \mathbf{g}_0{}'$;
2.2. $\delta(Y')J = J\delta(Y'), \forall Y' \in \mathbf{g}_0{}'$.

Obviously, direct sums of LCR-structures are LCR-structures: in fact, they correspond to $\delta = 0$. From the proposition it follows that if $\mathbf{h}_0$ is endowed with a complex structure and if $\delta(X)$ is holomorphic, $\mathbf{h}_0 \oplus_\delta \mathbf{g}_0$ supports a LCR-structure, where $\mathbf{g}_0$ is a generic real Lie algebra. That is the case of noncompact semisimple Lie algebras where $\mathbf{g}_0$ is the sum of the real factors and $\mathbf{h}_0$ is the sum of the Cartan-classified ones. A basic example is given by a reductive Lie algebra. In fact, in that case the algebra is the direct sum of its center and of a semisimple Lie subalgebra. So, an LCR-structure is direct sum of an abelian LCR-structure with a semisimple one. Such a situation bring us to consider Levi–Mal'cev decomposition. In this decomposition one factor is semisimple while the other one is solvable, a little more generic than abelian. Its study is the object of the following section.

4. Levi–Mal'cev decomposition

The tool of this section is the study of LCR-structures $\Gamma_0 = (\mathbf{g}_0, \mathbf{p}, J)$ on a generic Lie algebra $\mathbf{g}_0$ decomposed, following Levi–Mal'cev, as $\mathbf{g}_0 = \mathbf{r} \oplus_{\mathrm{ad}} \mathbf{s}$, where $\mathbf{r}$ is the solvable radical and $\mathbf{s}$ is a Levi subalgebra. Recall that the Levi subalgebra is semisimple.

Due to the Lemma in Section 3, one decomposes $\mathbf{p}$ as $\mathbf{p} = \mathbf{p_r} \oplus \mathbf{p_s}$, where $\mathbf{p_r}$ and $\mathbf{p_s}$ satisfy the following relations

1.1. $[\mathbf{p_r}, \mathbf{r}] \subset \mathbf{p_r}$,
1.2. $[\mathbf{p_s}, \mathbf{s}] \subset \mathbf{p_s}$,
1.3. $[\mathbf{p_s}, \mathbf{r}] \subset \mathbf{p_r}$,
1.4. $[\mathbf{p_r}, \mathbf{s}] \subset \mathbf{p_r}$;

1.1 (resp. 1.2) means that $\mathbf{p_r}$ (resp. $\mathbf{p_s}$) is an ideal in $\mathbf{r}$ (resp. $\mathbf{s}$); 1.3 and 1.4 coincide respectively with the letters (b) and (c) of the Lemma.

The second characterizing property of an LCR-structure is $J^2 = -\mathrm{id}$. Just computing the square of $J = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, we obtain

2.1. $A^2 + BC = -I$,
2.2. $AB + BD = 0$,
2.3. $DC + CA = 0$,
2.4. $D^2 + CB = -I$.

Finally, let $\mathrm{ad}_{(X,X')}$ be a CR-derivation. That means $(A[U, V] + A[U, Y] + A[X, V] + B[X, Y], C[U, V] + C[U, Y] + C[X, V] + D[X, Y]) = ([U, AV + BY] + [U, CV + DY], +[X, AV + BY], [X, CV + DY])$. In particular, one may consider the cases given by the conditions $X = Y = 0$, $X' = Y' = 0$, $X = Y' = 0$ and $X' = Y = 0$. The corresponding equations are

3.1. $A[U, V] = [U, AV] + [U, CV]$,
3.2. $A[X, V] = [X, AV]$,
3.3. $A[U, Y] = [U, BY] + [U, DY]$,
3.4. $B[X, Y] = [X, BY]$,
3.5. $C[U, V] = 0$,
3.6. $C[U, Y] = 0$,
3.7. $C[X, V] = [X, CV]$,
3.8. $D[X, Y] = [X, DY]$.

Proposition. *The equations* 1.1–1.4, 2.1–2.4, 3.1–3.8 *define a necessary and sufficient condition in order that* $\Gamma_0 \oplus_\delta \Gamma'_0$ *be an LCR-structure.*

The above Proposition is just a "translation" of the LCR-structure's definition on a semidirect sum. Now we have to study the role played by the solvability af $\mathbf{r}$ and the semisemplicity of $\mathbf{s}$. In this context, their most important consequence is the

Property. *The matrices* B *and* C *vanish.*

Proof. Since $C[X, V] = [X, CV]$, $C(\mathbf{p_r})$ is an ideal in $\mathbf{s}$. But $[CV, CV_1] = C[CV, V_1] = 0$, so $C(\mathbf{p_r})$ is abelian. Consequently $C(\mathbf{p_r})$ vanishes.

It is a classical fact that every ideal and every quotient of semisimple algebras are semisimple [10], moreover $\mathbf{p_s}/\ker B$ is semisimple. Otherwise, $\mathbf{r}$ solvable implies that every subspace $\mathbf{t} \subset \mathbf{r}$ verifies $\mathbf{D}^n\mathbf{t} = 0$. So $B(\mathbf{p_s})$ does. As linear spaces, we have that $\mathbf{p_s}/\ker B \simeq B(\mathbf{p_s})$, via the isomorphism $jX^+ = BX$, where $X^+ = X + \ker B \in \mathbf{p_s}/\ker B$.

Let us compute $[jX^+, jY^+]$: first of all, $\forall X, Y \in \mathbf{p_s}$, $[BX, BY] = A[BX, Y] - [BX, DY] = AB[X, Y] - B[X, DY] = -BD[X, Y] - B[X, DY] = -2BD[X, Y]$. By (2.2), you see that D sends $\ker B$ in $\ker B$ and hence $[jX^+, jY^+] = -2j(D[X, Y])^+$. So, we can conclude that $\mathbf{D}^n(\mathbf{p_s}/\ker B) = 0$, since $B(\mathbf{p_s})$ does. But the fact that $\mathbf{p_s}/\ker B$ is semisimple implies that its solvable radical vanishes. So $\mathbf{p_s} = \ker B$. $\square$

Remark. The condition "$C = 0$" is true even in a more general case. The only required hypothesis is the semisemplicity of $\mathbf{s}$. So if you take a semidirect sum with second factor $\mathbf{g}'_0$ semisimple, an LCR-structure takes the form $(\mathbf{g}_0 \oplus_\delta \mathbf{g}'_0, \mathbf{p} \oplus \mathbf{p}', \begin{pmatrix} A & B \\ 0 & D \end{pmatrix})$.

The Property shows that J takes the particular form $J = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$. Hence, our list of equations simplify itself to

1.1. $[\mathbf{p_r}, \mathbf{r}] \subset \mathbf{p_r}$,
1.2. $[\mathbf{p_s}, \mathbf{s}] \subset \mathbf{p_s}$,
1.3. $[\mathbf{p_s}, \mathbf{r}] \subset \mathbf{p_r}$,
1.4. $[\mathbf{p_r}, \mathbf{s}] \subset \mathbf{p_r}$,
2.1′. $A^2 = -I$,
2.4′. $D^2 = -I$.
3.1′. $A[U, V] = [U, AV]$,
3.2. $A[X, V] = [X, AV]$,
3.3′. $A[U, Y] = [U, DY]$,
3.8. $D[X, Y] = [X, DY]$.

1.1, 2.1′ and 3.1′ say that $R = (\mathbf{r}, \mathbf{p_r}, A)$ is a LCR; 1.2, 2.4′ and 3.8′ say that $S = (\mathbf{s}, \mathbf{p_s}, D)$ is an LCR-structure; finally 1.3, 3.3′, 1.4 and 3.2 correspond to 1.1, 1.2, 2.1 and 2.2 of Theorem 3, respectively. So we can conclude with

Theorem 4. *Let* $\mathbf{g}_0 = \mathbf{r} \oplus_{\mathrm{ad}} \mathbf{s}$ *be a real Lie algebra. Suppose* $\Gamma_0 = (\mathbf{g}_0, \mathbf{p}, J)$ *is an LCR-structure; then R and S are, too; and Γ_0 is their semidirect sum by* ad. *Vice versa, if one considers two LCR-structures* $R = (\mathbf{r}, \mathbf{p_r}, A)$ *and* $S = (\mathbf{s}, \mathbf{p_s}, D)$ *which satisfy*

1.3. $[\mathbf{p_s}, \mathbf{r}] \subset \mathbf{p_r}$,
1.4. $[\mathbf{p_r}, \mathbf{s}] \subset \mathbf{p_r}$,
3.2. $A[X, V] = [X, AV]$,
3.3′. $A[U, Y] = [U, DY]$,

their semidirect sum by ad *is an LCR-structure on* $\mathbf{g}_0$.

5. Examples: LCR on low-dimensional real Lie algebra

In this section we conclude the paper describing LCR-structures for low dimensional Lie algebras $\mathbf{g}_0$. First of all, recall that there exist just two different bidimensional Lie algebras: the abelian one and the Lie algebra $\mathbf{h}_0$ of the matrices $\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}$, which is solvable. Both of them are endowed with the complex structure given by the "multiplication by i."

If one wants to consider LCR-structures on real Lie algebras $\mathbf{g}_0$ which are not complex structures, it must be $\dim \mathbf{g}_0 \geqslant 3$. Let us start with $\dim \mathbf{g}_0 = 3$. Such Lie algebras are completely classified in [9]. The classification makes use of the map $\varphi : \mathbf{g}_0 \to \mathbf{R} : X \mapsto \mathrm{tr}(\mathrm{ad}_X)$. Since $\mathrm{tr}([\mathrm{ad}_X, \mathrm{ad}_Y]) = 0$, φ is a Lie homomorphism. The kernel $\mathbf{u} := \ker \varphi$ is an ideal called *unimodular kernel*; $\mathbf{g}_0$ is said *unimodular* if $\mathbf{g}_0 = \mathbf{u}$. An important result is given by the

Lemma. *Let* $\mathbf{g}_0$ *be an unimodular 3-dimensional Lie algebra endowed with a scalar product. Then there exists an orthonormal base* (E_1, E_2, E_3) *such that*

1. $[E_2, E_3] = \lambda_1 E_1$, $[E_3, E_1] = \lambda_2 E_2$ *and* $[E_1, E_2] = \lambda_3 E_3$;
2. $B(X, Y) = -2(\lambda_2\lambda_3 X^1 Y^1 + \lambda_1\lambda_3 X^2 Y^2 + \lambda_1\lambda_2 X^3 Y^3)$, *where* $X = X^i E_i$, $Y = Y^i E_i$.

For a proof, see [5].

The 3-dimensional Lie algebras are classified by the following cases

1. $\lambda_1 = \lambda_2 = \lambda_3 = 0$,
2. $\lambda_1 \neq 0, \lambda_2 = \lambda_3 = 0$,
3. $\lambda_1\lambda_2 \neq 0, \lambda_3 = 0$,
4. $\lambda_1\lambda_2\lambda_3 \neq 0$.

Case 1. $\mathbf{g}_0$ is abelian and isomorphic to $\mathbb{R}^3$. Each plane supports an LCR structure: in fact, let $\mathbf{p} = L(X, Y)$ a fixed plane; a structure as desired is given by $J(X, Y) = (-Y, X)$.

Case 2. The Lie product is described by $[E_2, E_3] = \lambda_1 E_1$, $[E_3, E_1] = [E_1, E_2] = 0$. The planes $\mathbf{p}_2 = L(E_1, E_3)$ and $\mathbf{p}_3 = L(E_1, E_2)$ are abelian ideals endowed with the LCR structures $J_2(E_1, E_3) = (-E_3, E_1)$ and $J_3(E_1, E_2) = (-E_2, E_1)$.

Case 3. As in the case before, the plane $\mathbf{p}_3 = L(E_1, E_2)$ is an abelian ideal endowed with the structure $J_3(E_1, E_2) = (-E_2, E_1)$.

Case 4. B is nondegenerate, i.e. $\mathbf{g}_0$ is semisimple. But 3-dimensional semisimple Lie algebras are simple. Hence $\mathbf{g}_0$ has no nontrivial ideals. So there are no LCR-structures on such a $\mathbf{g}_0$. A deeper analysis shows that if all the λ_i are positive $\mathbf{g}_0$ is isomorphic to $\mathbf{su}(2)$; while if one of them is negative it is isomorphic to $\mathbf{sl}(2, \mathbb{R})$. In both the cases $\mathbf{g}_0$ is a real form (compact or not) for $\mathbf{sl}(2, \mathbb{C})$.

The last case is when $\mathbf{g}_0$ is not unimodular. Which means that φ is a nonvanishing real linear form. So its kernel $\mathbf{u}$ is an abelian 2-dimensional ideal.

Summarizing all the case, one obtains that a 3-dimensional real Lie algebra $\mathbf{g}_0$ either is a (simple) real form of $\mathbf{sl}(2, \mathbb{C})$ either is endowed with (at least) one LCR-structure given on a 2-dimensional abelian ideal.

The study of LCR-structures on 2- and 3-dimensional Lie algebras makes easy the classification on 5-dimensional ones (remark that, if one considers the 4-dimensional case, the only non-solvable Lie algebra endowed with an LCR-structure is $\mathbb{R} \oplus \mathbf{s}_0$, where $\mathbf{s}_0$ is a real form of $\mathbf{sl}(2, \mathbb{C})$). Such a study is quite interesting since it makes use of Levi–Mal'cev decomposition as we have shown in Section 4. In the sequel, let $\dim \mathbf{g}_0 = 5$. Suppose that $\mathbf{g}_0$ is decomposed as $\mathbf{g}_0 = \mathbf{r}_0 \oplus_{\mathrm{ad}} \mathbf{s}_0$. Let us consider the dimension $\dim \mathbf{r}_0$.

When $\dim \mathbf{r}_0 = 0$, $\mathbf{g}_0$ is semisimple. Since there are no semisimple algebras of dimension 1 and 2, $\mathbf{g}_0$ may not have nonvanishing ideals. So $\mathbf{g}_0$ is simple and it has no LCR-structures (cf. Table 1).

Let $\dim \mathbf{r}_0 = 1$. Then $\mathbf{r}_0 = \mathbb{R}$ is abelian and $\mathbf{s}_0$ is simple. Any LCR-structure, if it is, is in $\mathbf{s}_0$. But, $\mathbf{s}_0$ does not contain ideals. So $\mathbf{g}_0$ has no LCR-structures.

In the case $\dim \mathbf{r}_0 = 2$, $\mathbf{r}_0$ either is abelian or it is the solvable algebra $\mathbf{h}_0$. The corresponding Levi-subalgebra $\mathbf{s}_0$ is simple and coincides either with $\mathbf{su}(2)$ or with $\mathbf{sl}(2, \mathbf{R})$. Even in this case, $\mathbf{s}_0$ does not admit LCR-structures. The only one is given by the solvable ideal $\mathbf{r}_0$ endowed with an endomorphism of the form J_A.

The cases $\dim \mathbf{r}_0 = 3, 4$ cannot occur, since $\mathbf{s}_0$ should be 2- or 1-dimensional. The last case is $\dim \mathbf{r}_0 = 5$. Then $\mathbf{g}_0$ is solvable and it admits LCR-structures on all its 2- and 4-dimensional ideals.

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